Introduction to Welfare and Equilibrium

Partial Equilibrium - looks at single market
General Equilibrium - simultaneous equilibrium in all markets; markets are linked in a system so if one market is out of equilibrium, others can't be in equilibrium... can't understand single market without studying interrelated markets
- **Vertical Interaction** - hierarchy of markets; final good drives market for intermediate goods which drive markets for raw materials, but the interaction goes the other way too (e.g., steel shortage affects market for cars)
- **Horizontal Interaction** - income effects; substitutes/complements

When is Partial Eq. OK - Cobb-Douglas preferences (e.g., $u = x_1^{\alpha} x_2^{\beta}$) which result in independent demands: $x_i = \alpha_i I / P_i$ (there is no income effect between markets)

Existence - with single market it's easy to find market clearing price (equilibrium), but with multiple markets it's more difficult
- **Valras** - said to look for solution to simultaneous equations (i.e., supply = demand in each market); need same number of prices (unknowns) as markets (equations)... although this is correct, it doesn't guarantee a solution (possibility of redundant equations); mathematical tools Valras needed weren't developed yet
- **Arrow, Debreu, & McKenzie** - came up with ways to determine existence in 1940s-50s
  - Fixed Point Theorem - mathematical tools that made it possible
- **Why Check Existence** - logical check on model we're dealing with; never really study full economy so we make simplifications; in order to make sure assumptions are consistent and results aren't absurd
  - **Assumptions** - if we assume A and ~A, then we can prove anything
  - **Results** - makes so sense talking about equilibrium it if doesn't exist
- **Methods** - several ways to show existence
  1. **Formal Proof** - show that equilibrium also exists
  2. **Special Cases** - if only interested in specific examples, only show equilibrium exists in those circumstances, then write "when equilibrium exists..."; it's nice to know when "when" is, but that could be left for someone else to show once you've at least shown that sometimes there is equilibrium

Allocation - assignment of commodities to individuals to consume and inputs to producers
- **Feasible Allocation** - could actually be produced given technology and resource limitations (e.g., can't have firm producing 20 units with zero inputs)

Welfare Results -
- **Efficiency** - in partial equilibrium, efficiency is defined as maximum of sum of consumer and producer surplus
- **Pareto Optimality** - this is the same as efficiency for partial equilibrium, but not in general equilibrium (i.e., taking economy as a whole); an allocation is Pareto optimal if we can't find another allocation that makes every consumer better off
  - **Single Person Definition** - some authors use definition that we can't make one consumer better off without someone else being worse off... given divisible commodity like money, these are equivalent definitions because if you can make one person better off, you can make everyone better off
  - **Negative Definition** - if it is not possible to change in the allocation without hurting at least one person, then the allocation is Pareto optimal

P.O. vs. Efficiency - P.O. focuses only on consumers vs. "efficiency" which looks at producer and consumer surplus; technically, the producer surplus is a proxy for other markets

Production - only enters in determining feasibility; captured indirectly through consumers who own firms (firm's profits become their incomes which leads to consumption)

First Fundamental Theorem of Welfare Economics - outcome of competitive economy is Pareto optimal

Second Fundamental Theorem of Welfare Economics - look at all Pareto optimal allocations; each can be achieved using competitive equilibrium given different endowments (initial allocations); there is no unique P.O. outcome (e.g., Slutsky has everything; can't change that allocation without making Slutsky worse off ∴ it's a P.O. allocation); also called t

Equity - some economists say it's religious, ethical, or political question; it's not our job, but given a notion of equity, we can determine if an allocation is equitable

Unbiasedness Theorem - 2nd FTWE is said to be unbiased because inequity results form the initial allocation, not from the competitive market; ∴ economists argue that taxes or redistribution system should deal with equity issues, not prices because competitive market (i.e., prices) doesn't bias outcome in favor of particular group (results are based on initial allocation)

Redistribution - leads to incentive problem (e.g., income tax could cause people to work less); there's a tradeoff between equity and efficiency

Core - allocations that no coalition of any size can (or will) block

Theorem - as number of people increases, the set of core allocations declines; at the limit the set of core allocations converges to the competitive equilibrium

Stability - if economy is not at equilibrium will it move to equilibrium?... we don't have good models of adjustment so this isn't studied much

Problems - there are three cases when 1st FTWE doesn't hold

Market Power - in partial equilibrium, we can show that a monopolist exercises market power so we don't achieve competitive equilibrium; in general equilibrium, however, there never really is a monopolist (e.g., even if firm is monopolist in automobiles, it still has to compete with bicycle makers; technically it competes with all others firms in trying to get consumer's money)

Externalities - actions affect others; we'll assume externalities don't exist for most of the course (although they are important); we'll only address them at the end

Distinguishing Feature - schools of economics disagree on importance of externalities (e.g., Chicago School thinks externalities are less important and less pervasive and the free market can get around them... other schools disagree)

Coase Theorem - private bargaining will eliminate externalities

Information - we assume complete information in the course; private information means bargaining can work, but we lose some P.O. allocations so 2nd FTWE may not hold... lose unbiasedness of competitive markets

Special Case - we'll start course with pure exchange economy (i.e., no production); consumers will trade based on their initial allocation; we'll add production later, but the results don't change (just get harder math)

Pareto Optimality

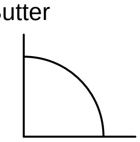
Institution Free - not based on supply, demand, price, consumer/producer surplus, etc. which requires assumptions about how goods are allocated

Goal - want a notion of efficiency that applies to all economies (i.e., institution free measure); want to be able to measure efficiency of economies that don't use price (e.g., socialism; no price means we can't use consumer and producer surplus)

- **Roommates** - don't use markets (prices); use negotiation and bargaining, but we still want to be able to determine if their actions are efficient

Notation -

- **Consumers** - represent with i = 1, 2, ..., n
- **Bundle of Consumption Goods** - $\mathbf{x}^i \in R^k$ ... k commodities
  - **Note:** real numbers not limited by sign (could be < = or > 0); a negative consumption good is what consumers pay to acquire the other goods (e.g., labor); standard notation for all consumption goods being ≥ 0 is $\mathbf{x}^i \in \Omega^k$ (limited to first quadrant)
- **Utility Function** - $u^i(\mathbf{x}^i)$ ... <u>assumption</u>: utility only based on own consumption (i.e., no externalities where utility is higher or lower based on other people's consumption)
- **Firms** - represented with j = 1, 2, ..., m
- **Technology Set** - Y^j ... same properties we covered (quickly) in micro:
  - [1] Nonempty: some $\mathbf{y} \in Y$ with $\mathbf{y} \neq \mathbf{0}$
  - [2] Closed
  - [3] Inactivity: $\mathbf{0} \in Y$
  - [4] No Free Lunch: $\mathbf{y} \geq \mathbf{0}$ & $\mathbf{y} \neq \mathbf{0} \Rightarrow \mathbf{y} \notin Y$ (can't have all outputs with no inputs)
  - [5] Free Disposal: $\mathbf{y} \in Y$ & $\mathbf{y}' \leq \mathbf{y} \Rightarrow \mathbf{y}' \in Y$ (monotonicity; produce less (or same) with more)
  - [6] Irreversibility: $\mathbf{y} \in Y$ & $\mathbf{y} \neq 0 \Rightarrow -\mathbf{y} \notin Y$ (will be loss if we reverse production process)
  - [7] Convex $\Rightarrow$ no increasing returns to scale
- **Aggregate Endowment Vector** - $\mathbf{E}$; amount of each commodity available (not assigning ownership right now because that's an institutional feature)
- **Allocation** - vector that tells what every consumer gets and every firm does (< 0 for inputs; > 0 for outputs): $(\mathbf{x},\mathbf{y}) = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n, \mathbf{y}^1, \mathbf{y}^2, \ldots \mathbf{y}^m)$ ... each sub-vector is k x 1
  - **Very Big** - for U.S. consider n = 300M and m at least 1M... that's a big allocation matrix!
- **Feasible Allocation** - one economy can actually achieve; 3 criteria:
  1) **Technologically Producible** - $\mathbf{y}^j \in Y^j \ \forall j$ = 1, 2, ..., m
  2) **Allocation is Attainable** - production within resource limits
     - **Production Possibilities Frontier (PPF)** - combines (1) and (2)
  3) **Budget Constraint** - what consumers get is either produced or initially available (i.e., consumption has to be within the PDF):



$$\sum_{i=1}^{n} \mathbf{x}^i \leq \sum_{j=1}^{m} \mathbf{y}^j + \mathbf{E}$$

Pareto Optimal (P.O.) - a feasible allocation $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ is Pareto optimal if there does not exist any other feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ with $u^i(\hat{\mathbf{x}}^i) \geq u^i(\overline{\mathbf{x}}^i) \ \forall i$ = 1, 2, ..., n with at least one strict inequality (i.e., where everybody is at least as well off and some person is better off)

- **Note:** this is an institution free definition... didn't say how allocation is determined
- **Pareto Improvement** - the feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a Pareto improvement over the feasible allocation $(\overline{\mathbf{x}}, \overline{\mathbf{y}})$ if $u^i(\hat{\mathbf{x}}^i) \geq u^i(\overline{\mathbf{x}}^i) \ \forall i$ = 1, 2, ..., n with some strict inequalities

Alternate Definition - $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is Pareto optimal if there are no Pareto improvements

Debate - if economy is Pareto optimal, it's in conflict because in order for anyone to do better, he has to hurt someone else; people entering admin/political jobs want the system to not be Pareto optimal so they can make changes that make everyone happy

Preferences - to keep the definition institution free, we need preferences to be a certain way (e.g., can't use amount person gives to charity as part of utility without addressing institution)

Determining P.O. - usually easier to check first and second order conditions rather than doing pair wise comparison of all feasible allocations... think of number of pairs (that's not a practical way to do it)

Pareto Problem - $\max_{(\mathbf{x},\mathbf{y})} u^1(\mathbf{x}^1)$ s.t. $(\mathbf{x},\mathbf{y})$ feasible... solution is to give everything to person 1 so we need the Pareto Constraints to ensure others have min level of utility

Pareto Constraints - $u^i(\mathbf{x}^i) \geq \bar{u}^i$, i = 2, 3, ..., n

Section 4 - we'll replace the Pareto constraint with weighted sums in the objective function to make the problem easier to solve: $\max_{(\mathbf{x},\mathbf{y})} \sum_{i=1}^{n} \alpha_i u^i(\mathbf{x}^i)$ s.t. $(\mathbf{x},\mathbf{y})$ feasible

Debate - this method is simpler, but not everyone agrees that it's the same

Information Constraints - another method is to replace the Pareto constraints with incentive compatibility constraints

Finding all P.O. Allocations - if using Pareto constraints, change values of $\bar{u}^i$; if using the weighted sums, change values of α_i

Politics

Notion of Equilibrium - "allocation" can't be overturned (as seen in this example)

Relative Majority Voting - method of making political decisions; assume several alternatives are available and society has to choose; e.g., with 2 alternatives A and B; 3 choices:

Some individuals prefer A: $u^i(A) > u^i(B)$; the number who prefer A is N_A

Some individuals prefer B: $u^j(B) > u^j(A)$; the number who prefer B is N_B

Some individuals are indifferent: $u^k(A) = u^k(B)$; the # who are indifferent is N_I

Direct Preferences - assume people vote their direct preferences (vs. strategic voting where individual may prefer A to B, but votes for B because B is more likely to defeat C)

Relative Majority - A defeats B if $N_A > N_B$... relative instead of absolute because N_I people don't count

Relative Majority Equilibrium (RME) - x is RME if there is no other feasible alternative y which defeats x

Theorem 1 - every RME is Pareto optimal

Proof: assume x is RME and is not Pareto optimal

Since it's not P.O., there must exist an alternative y with $u^i(y) \geq u^i(x)$ $\forall\ i$ and at least 1 individual with strict inequality

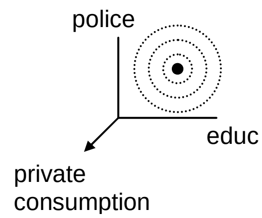
That means the number who prefer y to x is at least 1 (i.e., $N_y \geq 1$) and the number of people who prefer x to y is zero (i.e., $N_x = 0$)

Which means y defeats x by relative majority voting so x is not an RME... contradiction $\therefore$ if x is an RME it must be P.O.

Note: This theorem didn't say anything about the space of alternatives

Specific Case - assume compact, convex set of alternatives and three individuals who have satiated indifference curves each with a different ideal (or bliss) point

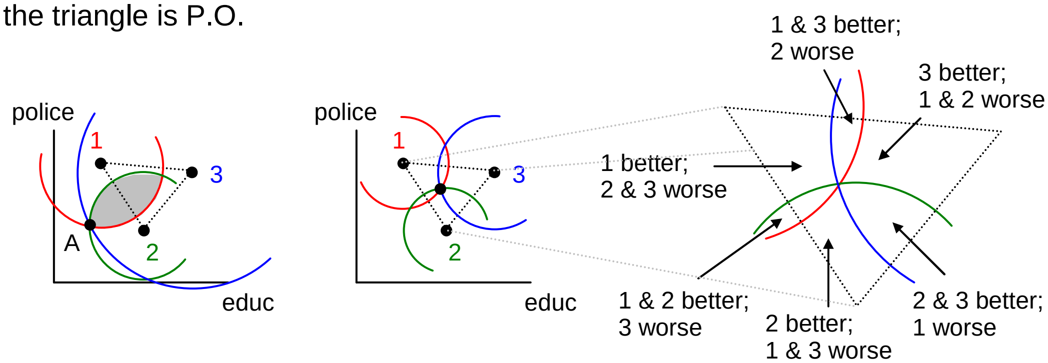
Satiated Preferences - individuals have a bliss point (utility increases as indifference curves converge on the bliss point); this is not necessarily a violation of well behaved preferences because we could just be embedding another factor and looking at the projection of preferences (e.g., education vs. police service with private consumption on third axis; more of both is better, but at cost of higher taxes [lower consumption] so there's a bliss point in educ-police space)



Find P.O. Alternatives - we'll assume indifference curves are circles so we get straight lines when we look at the tangencies between any two individuals' indifference curves (makes drawing the picture easier)

Look at points outside triangle... shaded area shows points where everyone is better off so points outside triangle are not P.O.

Look at points inside (and on) triangle... nothing is a Pareto improvement ∴ every point in the triangle is P.O.



Theorem 2 - every RME for this situation is not P.O. (i.e., there is no RME in the triangle)

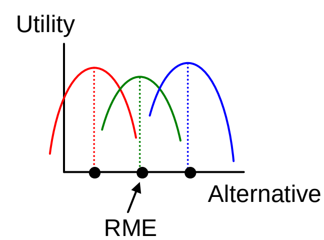
<u>Proof</u>: look at contrapositive: a P.O. is not an RME

Every point in the triangle is a P.O. (which we just showed)

Looking at the same diagram, there are clearly alternatives preferred by 2 of the 3 individuals so that point will not be an equilibrium with relative majority voting (which is what we had to show)

Problem - why does this theorem contradict the previous theorem?

In this situation no RME exists... it only exists with 1 dimensional alternative (in that case it's the median of ideal points as shown in the picture; plotting utility in vertical axis so these are <u>not</u> indifference curves)



Possible Solution - vote separately on education and police; that eliminates the existence problem (but now violating presumption of first theorem so the RME is not guaranteed to be P.O.)

Note: in the case above, it will be P.O., but if the indifference curves are not perfect circles there's no guarantee that the RME from voting separately would be P.O.

Point - we care if equilibrium exists; the first theorem is worthless if RME doesn't exist; correct working would be: "if RME exists, then the RME is P.O."

Simple Models of Exchange and Production

Beliefs - agents (players in game, producers & consumers in economy) have to make decisions based on beliefs

Rational - belief can be anything as long as evidence doesn't contradict it; if belief is contradicted by observation and agent still holds on to belief, he is nonsensical (irrational)

Competitive Market - all agents believe that they can sell (or buy) as much or as little as they want without changing market price; looking at downward sloping demand and upward sloping supply curves, this belief can't be considered rational unless market is at equilibrium (i.e., nobody wants to change so $\mathbf{q}$* is fixed)... since nobody wants to change, the belief that agents can buy or sell more isn't contradicted by evidence ∴ nobody has to change their beliefs

Equilibrium - beliefs don't change

Competitive Equilibrium - looking at static (one period) model; will now make assumptions about institutions (which we didn't do to define Pareto optimal); 3 conditions

Max Profit - every firm is maximizing profits, taking prices as given (subject to feasibility; i.e., within technology set)

Max Utility - every consumer is maximizing utility, taking prices as given (subject to feasibility; i.e., within budget constraint)

Market Clearing - everybody takes same prices as given and at those prices supply equals demand for every commodity

Formally - CE is a price vector $\mathbf{p}^*$, a set of production vectors $\mathbf{y}^{j*}$ ($j = 1, \ldots, k$) and a set of consumption vectors $\mathbf{x}^{i*}$ $(i = 1, \ldots, m)$

Complete Markets - $(\mathbf{p}^*, \mathbf{x}^*, \mathbf{y}^*)$ ... we'll <u>assume</u> complete markets, meaning there is a price for each commodity; this is not always a good assumption (e.g., in a dynamic model, we need time-dated commodities and prices for future markets don't always exist; another example is contingent commodities [utilities are based on states of nature]); 2 methods:

Vectors - 1 x n, 1 x $n{\cdot}m$, and 1 x $n{\cdot}k$, respectively... makes math easier

Matrices - all vectors have the same dimensions, 1 x n, m x n and k x n, respectively... more intuitive

Producers - $\mathbf{y}^j$ is netput vector (i.e., negative elements are inputs to production; positive elements are commodities produced)

Feasible - $\mathbf{y}^{j*} \in Y^j$ (i.e., within firm j's technology set)

Max Profit - $\mathbf{p}^* \cdot \mathbf{y}^{j*} \geq \mathbf{p}^* \cdot \mathbf{y} \;\; \forall \;\; \mathbf{y} \in Y^j \quad (j = 1, \ldots, k)$

Consumers - $\mathbf{x}^i$ is similar to netput vector (i.e., negative elements are income [e.g., time, land]; positive elements are commodities consumed)

Feasible - $x^{i*} \in B^i(\mathbf{p}^*)$ (i.e., within consumer i's budget set)

Income - 3 types: $I^i(\mathbf{p}) = \mathbf{p} \cdot \boldsymbol{\omega}^i + \sum_{j=1}^{k} \theta^{ij} \boldsymbol{\pi}^j(\mathbf{p}) + T^i$

1. **Physical Endowment** - $\boldsymbol{\omega}^i$ (1 x n vector of how much of each commodity consumer i starts with)

Total Endowment - aggregate endowment to society is $\boldsymbol{\omega} = \sum_{i=1}^{n} \boldsymbol{\omega}^i$

2. **Profit Share** - assuming partnership (not limited liability corporations which would be more realistic); individual i gets $\sum_{j=1}^{k} \theta^{ij} \boldsymbol{\pi}^j(\mathbf{p}^*)$

   Firm Ownership - θ^{ij} = % for firm j that individual i owns (gets share of profit, but also has to pay share of expenses)

   Closed Economy - $\sum_{i=1}^{m} \theta^{ij} = 1$ (all firms 100% owned by consumers)

3. **Transfers** - individual I gets lump sum transfer from government, T^i; use monetary transfer because it's easier and more general than transferring a commodity (e.g., how do you transfer time?)

   Balanced Budget - $\sum_{i=1}^{m} T^i = 0$ ... if < 0 government runs surplus (i.e., collects more than it hands out); if > 0 government runs a deficit

Caution - don't double count income (e.g., if using time endowment, must have leisure commodity, but don't count labor because time - leisure = labor)

Budget Set - $B^i(\mathbf{p}) = \{\mathbf{x}^i : \mathbf{p} \cdot \mathbf{x}^i \leq I^i(\mathbf{p})\}$

Max Utility - $u^i(\mathbf{x}^i*) \geq u^i(\mathbf{x}) \;\; \forall \;\; \mathbf{x} \in B^i(\mathbf{p}^*) \quad (i = 1, \ldots, m)$

Market Clearing - demand = supply in all markets; $\sum_{i=1}^{m} \mathbf{x}^i* = \sum_{i=1}^{m} \boldsymbol{\omega}^i + \sum_{j=1}^{k} \mathbf{y}^j*$

amount consumed = amount available (endowment + production);

Net Consumption - can also write this condition: $\sum_{i=1}^{m} \left(\mathbf{x}^i * - \boldsymbol{\omega}^i\right) = \sum_{j=1}^{k} \mathbf{y}^j*$

Excess Supply - could use ≤ instead of = (i.e., don't consume everything); in order to hold with equality must have (a) $\mathbf{p} > \mathbf{0}$ (i.e., no free goods) with strict monotonicity of preferences (not realistic) or (b) satiated preferences and free disposal, or (c) some prices < 0 (we usually try to avoid this last one)

Summary -

Profit: $\mathbf{y}^j* \in Y^j$ and $\mathbf{p}^* \cdot \mathbf{y}^j* \geq \mathbf{p}^* \cdot \mathbf{y} \;\; \forall \;\; \mathbf{y} \in Y^j \quad (j = 1, \ldots, k)$

Utility: $x^i* \in B^i(\mathbf{p}^*)$ and $u^i(\mathbf{x}^i*) \geq u^i(\mathbf{x}) \;\; \forall \;\; \mathbf{x} \in B^i(\mathbf{p}^*) \quad (i = 1, \ldots, m)$

where $B^i(\mathbf{p}) = \{\mathbf{x}^i : \mathbf{p} \cdot \mathbf{x}^i \leq I^i(\mathbf{p})\}$ and $I^i(\mathbf{p}) = \mathbf{p} \cdot \boldsymbol{\omega}^i + \sum_{j=1}^{k} \theta^{ij} \boldsymbol{\pi}^j(\mathbf{p}) + T^i$

Clearing: $\sum_{i=1}^{m} \mathbf{x}^i* = \sum_{i=1}^{m} \boldsymbol{\omega}^i + \sum_{j=1}^{k} \mathbf{y}^j*$

Welfare Theorems (review)

1st Fundamental Theorem of Welfare Economics - any CE is PO

2nd Fundamental Theorem of Welfare Economics - given an PO allocation, $\exists$ $\mathbf{p}$ and $\mathbf{T}$ such that CE results in same allocation

Pure Exchange Economy - simpler proofs if we ignore production

Simplified CE - special case of previous 3 conditions; now only have $(\mathbf{p}^*, \mathbf{x}^*)$

Utility: $x^i{}^* \in B^i(\mathbf{p}^*)$ and $u^i(\mathbf{x}^i{}^*) \geq u^i(\mathbf{x}) \;\forall\; \mathbf{x} \in B^i(\mathbf{p}^*)$ $(i = 1, \ldots, m)$

where $B^i(\mathbf{p}) = \{\mathbf{x}^i : \mathbf{p} \cdot \mathbf{x}^i \leq I^i(\mathbf{p})\}$ and $I^i(\mathbf{p}) = \mathbf{p} \cdot \boldsymbol{\omega}^i + T^i$

Clearing: $\sum_{i=1}^{m} \mathbf{x}^i{}^* = \sum_{i=1}^{m} \boldsymbol{\omega}^i$

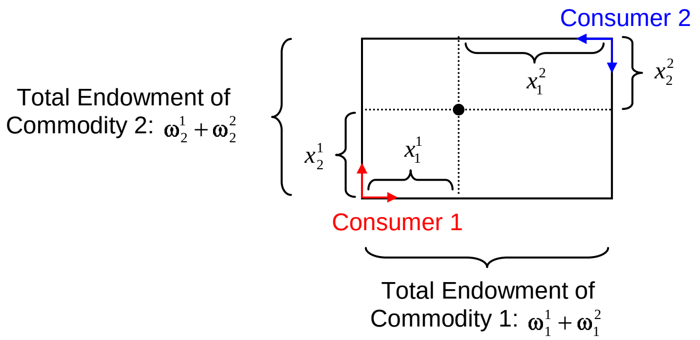
New Condition - $\mathbf{x}^i \geq \mathbf{0}$; consumers aren't allowed to supply inputs because this is pure exchange (i.e., no production)

Preference Ordering - recall from micro: complete, transitive, continuous, monotonicity (or local nonsatiation), and convexity; this is suppose to be easier than using utility representations

Pareto Optimal - allocation $\bar{\mathbf{x}}$ is PO if it is

1. **Feasible** - $\sum_{i=1}^{m} \bar{\mathbf{x}}^i{}^* = \sum_{i=1}^{m} \boldsymbol{\omega}^i$ and $\bar{\mathbf{x}}^i \geq \mathbf{0}$
2. **Max Utility** - there does not exist another feasible allocation $\mathbf{x}$ such that $\mathbf{x}^i \;\mathrm{R}^i\; \bar{\mathbf{x}}^i \;\forall\; i$ and $\mathbf{x}^j \;\mathrm{P}^i\; \bar{\mathbf{x}}^j$ for some j (i.e., at least one consumer is strictly better off and others are at least as well off)

Edgeworth Box - special case with 2 consumers and 2 commodities; points in box are feasible allocations with no waste



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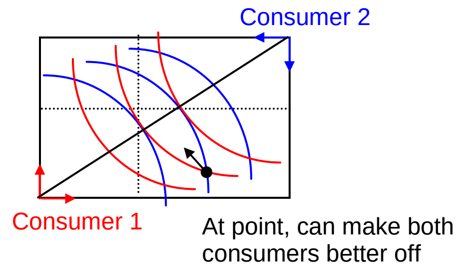
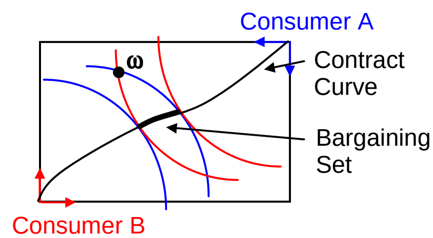


PO allocations are points where indifference curves are tangent

Contract Curve - locus of all PO allocations; 1st FTWE says if we find a CE it'll be on the contract curve

Bargaining Set - given endowment $\boldsymbol{\omega}$ (adjusted by transfers), the CE must lie between the indifference curves through $\boldsymbol{\omega}$; that segment is called the bargaining set



First Order Condition - tangency of indifference curves... $\mathrm{MRS}_{1,2}^{\mathrm{A}} = \mathrm{MRS}_{1,2}^{\mathrm{B}}$

Marginal Rate of Substitution - how change on 1 good affects change of another in order to keep utility constant $\left.\frac{dx_2}{dx_1}\right|^{u^{\mathrm{A}}} = -\frac{\partial u^{\mathrm{A}} / \partial x_1}{\partial u^{\mathrm{A}} / \partial x_2} = -\frac{u_1^{\mathrm{A}}}{u_2^{\mathrm{A}}}$

Proof 1 of 1st FTWE - assuming differentiability, we can use calculus to prove 1st FTWE

Costless Arbitrage - consumers can trade with each other at no cost; this is a usual assumption of completive markets; allows us to argue that both consumers pay the same price for the same good (i.e., $p_1^{\mathrm{A}} = p_1^{\mathrm{B}}$ and $p_2^{\mathrm{A}} = p_2^{\mathrm{B}}$)

Real World - not costless in real world, but concept still applies: consumers pay the same price for the same good up to the cost of arbitrage (e.g., if it costs $2 per transaction, then we'd expect the prices different consumers pay for the same good to be within $2 of each other)

Competitive Equilibrium - as mentioned on previous page, all we need to show for a competitive equilibrium is utility maximization for all consumers and market clearing

Maximization Problems -

Consumer A - $\max_{x_1^{\mathrm{A}}, x_2^{\mathrm{A}}} u^{\mathrm{A}}(x_1^{\mathrm{A}}, x_2^{\mathrm{A}})$ s.t. $p_1 x_1^{\mathrm{A}} + p_2 x_2^{\mathrm{A}} = p_1 \omega_1^{\mathrm{A}} + p_2 \omega_2^{\mathrm{A}} + T^{\mathrm{A}}$

Consumer B - $\max_{x_1^{\mathrm{B}}, x_2^{\mathrm{B}}} u^{\mathrm{B}}(x_1^{\mathrm{B}}, x_2^{\mathrm{B}})$ s.t. $p_1 x_1^{\mathrm{B}} + p_2 x_2^{\mathrm{B}} = p_1 \omega_1^{\mathrm{B}} + p_2 \omega_2^{\mathrm{B}} + T^{\mathrm{B}}$

Interior Solution - assume positive consumption of both commodities by both consumers; K-T Conditions are:

(1) $\frac{\partial u^{\mathrm{A}}}{\partial x_1^{\mathrm{A}}} - \lambda^{\mathrm{A}} p_1 = 0$ (4) $\frac{\partial u^{\mathrm{B}}}{\partial x_1^{\mathrm{B}}} - \lambda^{\mathrm{B}} p_1 = 0$

(2) $\frac{\partial u^{\mathrm{A}}}{\partial x_2^{\mathrm{A}}} - \lambda^{\mathrm{A}} p_2 = 0$ (5) $\frac{\partial u^{\mathrm{B}}}{\partial x_2^{\mathrm{B}}} - \lambda^{\mathrm{B}} p_2 = 0$

(3) $p_1 x_1^{\mathrm{A}} + p_2 x_2^{\mathrm{A}} - p_1 \omega_1^{\mathrm{A}} - p_2 \omega_2^{\mathrm{A}} - T^{\mathrm{A}} = 0$ (6) $p_1 x_1^{\mathrm{B}} + p_2 x_2^{\mathrm{B}} - p_1 \omega_1^{\mathrm{B}} - p_2 \omega_2^{\mathrm{B}} - T^{\mathrm{B}} = 0$

Market Clearing -

(7) $x_1^{\mathrm{A}} + x_1^{\mathrm{B}} = \omega_1^{\mathrm{A}} + \omega_1^{\mathrm{B}}$ (8) $x_2^{\mathrm{A}} + x_2^{\mathrm{B}} = \omega_2^{\mathrm{A}} + \omega_2^{\mathrm{B}}$

Prove PO - need to show that 8 equations above lead to Pareto optimal allocation (i.e., show indifference curves are tangent by showing MRS for each consumer is the same)

Solve for λ^{A} in (1) & (2): $\lambda^{\mathrm{A}} = \frac{\partial u^{\mathrm{A}} / \partial x_1^{\mathrm{A}}}{p_1} = \frac{\partial u^{\mathrm{A}} / \partial x_2^{\mathrm{A}}}{p_2} \Rightarrow \frac{\partial u^{\mathrm{A}} / \partial x_1^{\mathrm{A}}}{\partial u^{\mathrm{A}} / \partial x_2^{\mathrm{A}}} = \frac{p_1}{p_2}$

Solve for λ^{B} in (4) & (5): $\lambda^{\mathrm{B}} = \frac{\partial u^{\mathrm{B}} / \partial x_1^{\mathrm{B}}}{p_1} = \frac{\partial u^{\mathrm{B}} / \partial x_2^{\mathrm{B}}}{p_2} \Rightarrow \frac{\partial u^{\mathrm{B}} / \partial x_1^{\mathrm{B}}}{\partial u^{\mathrm{B}} / \partial x_2^{\mathrm{B}}} = \frac{p_1}{p_2}$

Putting these together we see that $\mathrm{MRS}_{1,2}^{\mathrm{A}} = \mathrm{MRS}_{1,2}^{\mathrm{B}}$ $\therefore$ CE (if it exists) is PO

Existence - add (3) and (6):

$p_1(x_1^{\mathrm{A}} + x_1^{\mathrm{B}}) + p_2(x_2^{\mathrm{A}} + x_2^{\mathrm{B}}) - p_1(\omega_1^{\mathrm{A}} + \omega_1^{\mathrm{B}}) - p_2(\omega_2^{\mathrm{A}} + \omega_2^{\mathrm{B}}) - (T^{\mathrm{A}} + T^{\mathrm{B}}) = 0$

Balanced Budget - balanced budget assumption says $T^{\mathrm{A}} + T^{\mathrm{B}} = 0$

Excess Demand - $\mathrm{ED}^i(p_1, p_2) = x_i^{\mathrm{A}} + x_i^{\mathrm{B}} - \omega_i^{\mathrm{A}} - \omega_i^{\mathrm{B}}$ ... depends on prices because amount of good demanded (x_i^{A} and x_i^{B}) depends on prices

Value of Excess Demand - $p_i \mathrm{ED}^i(p_1, p_2)$

Walras' Law - pronounced Val-ross; $\sum_i p_i \mathrm{ED}^i(p_1, p_2) = 0$

- **Interrelated Markets** - Walras' Law says markets are interrelated by budget/income concerns (e.g., if $\mathrm{ED}^1 > 0$ then $\mathrm{ED}^2 < 0$)
- **Always Holds** - Walras' Law always holds, not just in equilibrium, but if we assume equilibrium and assume prices are > 0, this gives us the market clearing conditions (7) and (8)

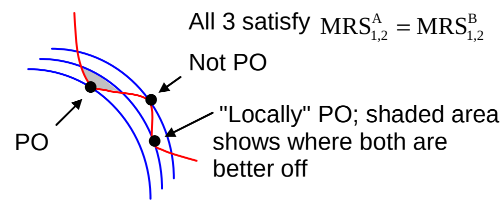
Second Order Conditions - technically we just proved first order conditions of CE and PO are the same; we didn't address the second order conditions to show that equilibrium exists; in this example (well behaved preferences [strictly convex indifference curves and strictly quasiconcave utility representations] with linear budget lines, we know equilibrium (i.e., max utility) exists

No Existence Example - assume $\omega_1^{\mathrm{B}} = 0$ (i.e., consumer A is monopolist for good 1);

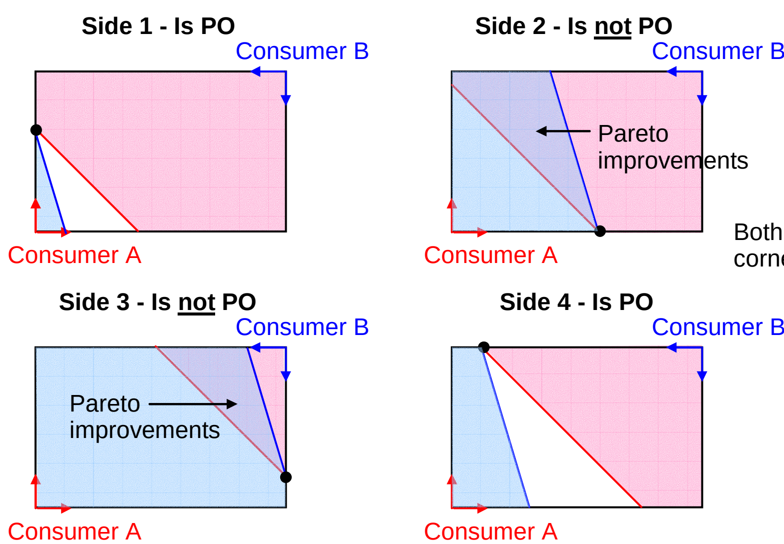
person A looks at $p_1^{\mathrm{A}}(x_1^{\mathrm{A}})$ ... equation (1) above changes: $\dfrac{\partial u^{\mathrm{A}}}{\partial x_1^{\mathrm{A}}} - \lambda^{\mathrm{A}}\left(p_1 + x_1^{\mathrm{A}}\dfrac{dp_1}{dx_1^{\mathrm{A}}}\right) = 0$,

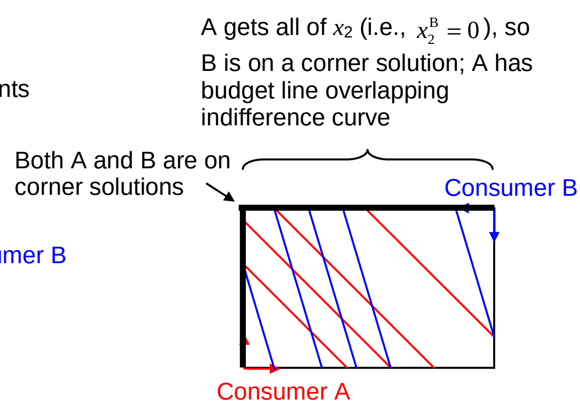
so we get $\mathrm{MRS}_{1,2}^{\mathrm{B}} = \dfrac{p_1}{p_2}$ but $\mathrm{MRS}_{1,2}^{\mathrm{A}} = \dfrac{p_1}{p_2} + \dfrac{x_1^{\mathrm{A}}}{p_2}\dfrac{dp_1}{dx_1^{\mathrm{A}}}$ ... ???

Convexity Assumption - if we drop the convexity assumption, we can find points that are not PO, but have $\mathrm{MRS}_{1,2}^{\mathrm{A}} = \mathrm{MRS}_{1,2}^{\mathrm{B}}$ $\therefore$ satisfying FOC and SOC locally does not guarantee a PO solution (although it will be locally PO)

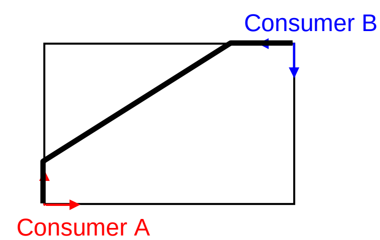


Corner Solutions - don't have to be so dramatic, but for simplicity let's consider perfect substitutes (indifference curves are straight lines)





Technicality - on side 1, consumer A in on corner solution and B has budget line overlapping his indifference curve; technically B is indifferent to all points on his indifference curve, but the point on the edge of the Edgeworth box is the only one that is consistent with equilibrium; ditto if we replace underlines with 4, B, A, A



Less Dramatic - don't need perfect substitutes; just need indifference curves to intercept the axes

Proof 2 of 1st FTWE - "more elegant"; using preferences instead of utility representations we can make fewer assumptions and the proof is "easier"

Assumptions - preferences are complete, transitive, and locally nonsatiated

Restate 1st FTWE - if $(\mathbf{p}^*, \mathbf{x}^*)$ is a competitive equilibrium, then $\mathbf{x}^*$ is Pareto optimal

Step 1 - deriving a lemma that will be used in the proof of 1st FTWE

$\mathbf{x}^i \ \mathrm{P}^i \ \mathbf{x}^i{}^*$ at $\mathbf{p}^* \Rightarrow \mathbf{p}^* \cdot x^i > \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$

Revealed Preference Argument - anything better than $\mathbf{x}^*$ must cost more or it would've been chosen instead of $\mathbf{x}^*$

$\mathbf{x}^i \ \mathrm{I}^i \ \mathbf{x}^i{}^*$ at $\mathbf{p}^* \Rightarrow \mathbf{p}^* \cdot x^i \geq \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$

Proof (by contradiction) - assume $\mathbf{p}^* \cdot \mathbf{x}^i < \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$

From local nonsatiation $\exists$ $\mathbf{y}^i$ "near" $\mathbf{x}^i$ with $\mathbf{y}^i \ \mathrm{P}^i \ \mathbf{x}^i$

Because we can get as close to $\mathbf{x}^i$ as we want, we can ensure $\mathbf{p}^* \cdot \mathbf{y}^i < \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$

From transitivity $\mathbf{y}^i \ \mathrm{P}^i \ \mathbf{x}^i{}^*$ which contradicts choosing $\mathbf{x}^i{}^*$ because there's a better bundle in the budget set $\therefore$ we must have $\mathbf{p}^* \cdot x^i \geq \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$

Step 2 - (by contradiction) assume $(\mathbf{p}^*, \mathbf{x}^*)$ is a competitive equilibrium, but not PO

No PO means $\exists$ a feasible allocation $\mathbf{x}$ such that $\mathbf{x}^i \ \mathrm{R}^i \ \mathbf{x}^i{}^* \ \forall \ i$ and $\mathbf{x}^j \ \mathrm{P}^j \ \mathbf{x}^j{}^*$ for some j (i.e., everyone at least as well off and at least one person is better off)

From Step 1 that means: $\mathbf{p}^* \cdot x^j > \mathbf{p}^* \cdot \boldsymbol{\omega}^j + T^j$ and $\mathbf{p}^* \cdot x^i \geq \mathbf{p}^* \cdot \boldsymbol{\omega}^i + T^i$ (for $i \neq j$)

Add these conditions across all consumers: $\mathbf{p}^* \cdot \sum_{i=1}^{m} \mathbf{x}^* > \mathbf{p}^* \cdot \sum_{i=1}^{m} \boldsymbol{\omega}^i + \sum_{i=1}^{m} T^i$

From balanced budget assumption $\sum_{i=1}^{m} T^i = 0$

Rewrite the inequality above: $\mathbf{p}^* \cdot \sum_{i=1}^{m} \left(\mathbf{x}^i{}^* - \boldsymbol{\omega}^i\right) > 0$

Assuming $\mathbf{p}^* > \mathbf{0}$, that means $\sum_{i=1}^{m} \left(\mathbf{x}^i{}^* - \boldsymbol{\omega}^i\right) > 0$ which violates budget so $\mathbf{x}$ is not feasible $\therefore$ $\mathbf{x}^*$ is PO

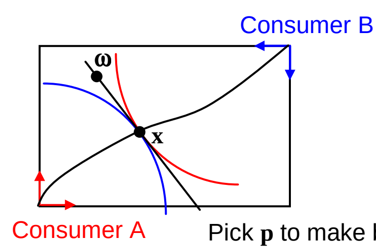
Note: proof is essentially the same if we incorporate production, but there are a few more mechanics to include profit shares

Existence - once again, we didn't look at the existence of CE; if we assume convexity and continuity of preferences, then we can prove the CE exists

Proof of 2nd FTWE - take any point on the contract curve (i.e. PO allocation); we can find $\mathbf{p}$ and $\boldsymbol{\omega}$ to make that point a CE

Informal Proof - set $\mathbf{p}$ so that the budget line is tangent to both indifference curves that are tangent at $\mathbf{x}$; can actually use any endowment vector as long as when we adjust it with transfer, we end up with $\boldsymbol{\omega}$ on the budget line

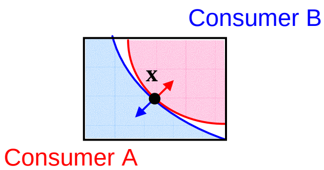
Note: we need global convexity of indifference curves for this argument to work

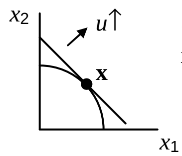


Pick $\mathbf{p}$ to make budget line tangent to both indifference curves

Problem Cases - couple of scenarios in which 2nd FTWE doesn't hold; we need to figure out what assumptions need to be made for it to hold (like we did for 1st FTWE)

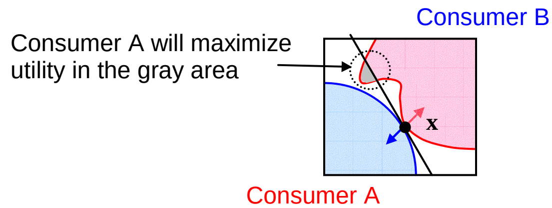
1. **Non-Convex Preferences** - given the indifference curves shown below, the allocation **x** is Pareto optimal, but it's not achievable as a competitive equilibrium... <u>problem</u>: consumer B's preferences are not convex:





Consumer B won't stay at allocation **x** if there is a linear budget constraint because he's not maximizing utility

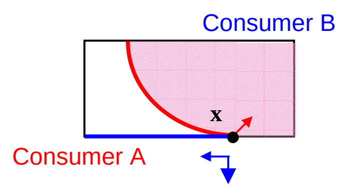
2. **Not Globally Convex Preference** - given indifference curves shown on the right, the allocation **x** is Pareto optimal, but it's not achievable as a competitive equilibrium... <u>problem</u>: consumer A's preferences are not globally convex



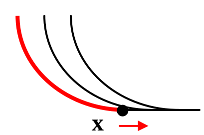
3. **Arrow's Exceptional Case** - actually, this is based on Arrow's Exceptional Case, but it adds the continuity problem; this is pretty specific so pay close attention:
   - i. $p_1 = 0$, $p_2 > 0$ (i.e., budget line is horizontal)
   - ii. endowment $\boldsymbol{\omega}$ on the boundary as shown below (i.e., consumer A owns none of good 2)
   - iii. Consumer A has preferences that intercept the x_1 axis with slope 0
   - iv. Consumer B has lexicographic preferences with good 2 as the primary (i.e. $\mathbf{x}' \succ \mathbf{x}$ if $x_2' > x_2$ or [$x_2' = x_2$ and $x_1' > x_1$])

   In this case, allocation **x** is Pareto optimal, but it's not achievable as a competitive equilibrium... <u>problem</u>: budget line goes through endowment and is horizontal ∴ can't have any point on this line as a competitive equilibrium (both consumers are better off with more x_1 which moves them away from **x**

   What's Wrong - actually, there are two problems in this case:
   (1) Endowment is on the <u>boundary</u>
   (2) Consumer B's preferences are <u>not continuous</u> (recall, lexicographic preferences are complete, transitive, monotonic, and convex, but not continuous)



Consumer B has lexicographic prefs; moving down (more $x2$) is always better; for given x_2, moving left (more x_1) is better



Consumer A's higher indiff. curves also hit the axis with slope zero, so his utility increases as $x_1\uparrow$ with $x_2 = 0$

Quasi-equilibrium - minimizing expenditure; it's the same as maximizing utility for interior solutions, but not at corners; consider second graph from Arrow's Exceptional Case above; with horizontal budget line, consumer A's utility is unbounded (can always get more x_1 if $p_1 = 0$), but allocation **x** is the expenditure minimization point for the level of utility shown by the red (thicker) indifference curve

Note: for lexicographic preferences, if the price of the less preferred good is zero, every point is a quasi-equilibrium (e.g., from above, since Consumer B prefers to move down first, a horizontal budget line would ensure that all of his consumption points are quasi-equilibria... min cost for the given level of utility; any other budget line would all him to give up some x_1 for x_2 which makes him better off and costs less)

Two-Step Proof - first we want to show that any PO point can be sustained as a quasi-equilibrium; then we'll determine what assumptions are necessary to guarantee a quasi-equilibrium is a competitive equilibrium

Restate 2nd FTWE - assume consumers have preferences which are complete, transitive, locally nonsatiated, and convex; let $\mathbf{x}^*$ be any PO allocation; ∃ a price vector $\mathbf{p}^* \neq \mathbf{0}$ (not all prices are zero) and income I^i for each individual with $\sum_{i=1}^{m} I^i = \sum_{i=1}^{m} \mathbf{p}^* \cdot \boldsymbol{\omega}^i$ (transfers embedded in the endowments) such that:

(i) if $\mathbf{x}^i \; \mathrm{P}^i \; \mathbf{x}^{i}*$ at $\mathbf{p}^* \Rightarrow \mathbf{p}^* \cdot x^i \geq I^i$ (i.e., any better vector costs at least as much)

(ii) $\sum_{i=1}^{m} \mathbf{x}^{i}* = \sum_{i=1}^{m} \boldsymbol{\omega}^i$ (i.e., demand = supply)

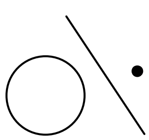
Note1: budget constraint is $I^i = \mathbf{p} \cdot \boldsymbol{\omega}^i + T^i$; given balanced budget, when we sum over all consumers the transfers go away (sum to zero) so we have the equation above... total income equals total value of endowments

Note2: in (i) we changed > to ≥ (compared to 1st FTWE); that's because then we were dealing with utility maximization; now we want to deal with expenditure minimization so we want to allow that there may be a better bundle that costs the same amount (we're not worried about that, just worried about weakly preferred bundles that cost less)

Supporting Math Stuff

Hyperplane - "plane" (i.e., linear surface in $n-1$ dimensions) in n dimensions (e.g., a line in 1d; a 2d flat surface (plane) in 3d)

Formally - in n dimensional space, let $\mathbf{p}$ be an n dimensional vector and r be a number, the set of all n dimensional vectors $\mathbf{x}$ such that $\mathbf{p} \cdot \mathbf{x} = r$ is a hyperplane (e.g., budget constraint is a hyperplane)

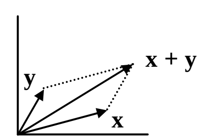


Separating Hyperplane Theorem - given a convex set and a point not in the convex set, we can draw a hyperplane so the point and convex set are on opposite sides of the hyperplane (the hyperplane is not necessarily unique)

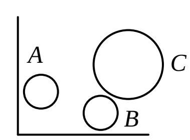
Formally - if A is a convex set and $\mathbf{x}$ is a point not in the closure of A, then ∃ $\mathbf{p}$ and r such that $\mathbf{p} \cdot \mathbf{x} > r$ and $\mathbf{p} \cdot \mathbf{y} < r \;\; \forall \;\; \mathbf{y} \in A$

(could use ≥ or ≤, but not both)

Summation of Vectors - $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \end{bmatrix}$



Summation of Sets - $C = A + B \neq A \cup B$; $\mathbf{z} \in C$ *iff* ∃ $\mathbf{x} \in A$ and $\mathbf{y} \in B$ such that $\mathbf{x} + \mathbf{y} = \mathbf{z}$



Theorem - if A and B are convex, then $C = A + B$ is convex

<u>Proof</u>:

Assume A and B are convex

Pick any $\mathbf{x}, \mathbf{x}' \in A$ and $\mathbf{y}, \mathbf{y}' \in B$

By definition, $\mathbf{z} = \mathbf{x} + \mathbf{y} \in C$ and $\mathbf{z}' = \mathbf{x}' + \mathbf{y}' \in C$

By convexity, $[\lambda \mathbf{x} + (1-\lambda)\mathbf{x}'] \in A$ and $[\lambda \mathbf{y} + (1-\lambda)\mathbf{y}'] \in B$ for $\lambda \in (0,1)$

By definition, $[\lambda\mathbf{x}+(1-\lambda)\mathbf{x}']+[\lambda\mathbf{y}+(1-\lambda)\mathbf{y}']\in C$

Rearrange terms, $[\lambda(\mathbf{x}+\mathbf{y})+(1-\lambda)(\mathbf{x}'+\mathbf{y}')]\in C$

Sub for the sums, $[\lambda\mathbf{z}+(1-\lambda)\mathbf{z}']\in C$

$\therefore$ C is convex

Preferred Set - $\mathrm{R}^{>i}(\mathbf{x}^i*)\equiv\{\mathbf{x}:\mathbf{x}\ \mathrm{P}^i\ \mathbf{x}^i*\}$

Pareto Optimal - new definition: $\mathbf{x}*$ is PO, if no feasible allocation $\mathbf{x}$ exists that is in $\mathrm{R}^{>i}(\mathbf{x}^i*)$ $\forall\ i$

Part 1 - if preferences are complete, convex, transitive, and locally nonsatiated, then any PO allocation $\mathbf{x}*$ can be sustained as a QE

A. $\mathrm{R}^{>i}(\mathbf{x}^i*)\equiv\{\mathbf{x}:\mathbf{x}\ \mathrm{P}^i\ \mathbf{x}^i*\}$ is convex

Proof: take any 2 bundles in $\mathrm{R}^{>i}(\mathbf{x}^i*)$, call them $\mathbf{z}$ and $\mathbf{w}$

By definition we know $\mathbf{z}\ \mathrm{P}^i\ \mathbf{x}^i*$ and $\mathbf{w}\ \mathrm{P}^i\ \mathbf{x}^i*$

Need to show that $\lambda\mathbf{z}+(1-\lambda)\mathbf{w}\ \mathbf{z}\ \mathrm{P}^i\ \mathbf{x}^i*$ (for $\lambda\in(0,1)$)

Because preferences are complete, either $\mathbf{z}\ \mathrm{R}^i\ \mathbf{w}$ or $\mathbf{w}\ \mathrm{R}^i\ \mathbf{z}$

Without loss of generality assume it's the first one (just a naming convention)

Because preferences are convex, $\mathbf{z}\ \mathrm{R}^i\ \mathbf{w}\Rightarrow\lambda\mathbf{z}+(1-\lambda)\mathbf{w}\ \mathrm{R}^i\ \mathbf{w}$ (for $\lambda\in(0,1)$)

Because preferences are transitive, $\lambda\mathbf{z}+(1-\lambda)\mathbf{w}\ \mathrm{P}^i\ \mathbf{x}^i*$

$\therefore$ $\mathrm{R}^{>i}(\mathbf{x}^i*)$ is convex

B. $\mathrm{R}^{>}(\mathbf{x}*)\equiv\sum_{i=1}^{m}\mathrm{R}^{>i}(\mathbf{x}^i*)$ is convex

Note: $\mathrm{R}^{>}(\mathbf{x}*)$ is set of *aggregate consumptions* in which there is some way to distribute to make everyone better off

Proof: already proved on previous page... sum of convex sets is a convex set

C. Aggregate endowment $\boldsymbol{\omega}\equiv\sum_{i=1}^{m}\boldsymbol{\omega}^i\notin\mathrm{R}^{>}(\mathbf{x}*)$ because $\mathbf{x}*$ is PO

Proof: (by contradiction) assume $\boldsymbol{\omega}\in\mathrm{R}^{>}(\mathbf{x}*)$

That means there is some aggregate consumption $\bar{\mathbf{x}}=\boldsymbol{\omega}$, where $\bar{\mathbf{x}}\in\mathrm{R}^{>}(\mathbf{x}*)$

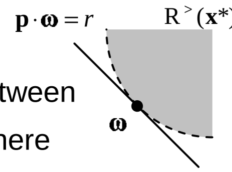
and $\bar{\mathbf{x}}=\sum_{i=1}^{m}\bar{\mathbf{x}}^i$ and $\bar{\mathbf{x}}^i\in R^{>i}(\mathbf{x}*)$ $\forall\ i$ ... don't necessarily know what each $\bar{\mathbf{x}}^i$ is, but they must by definition

Since $\bar{\mathbf{x}}=\boldsymbol{\omega}$, then $\mathbf{p}\cdot\bar{\mathbf{x}}=\mathbf{p}\cdot\boldsymbol{\omega}$, which means $\bar{\mathbf{x}}$ is feasible and strictly preferred by every consumer... that means $\mathbf{x}*$ is not PO

Simplifying Assumption - assuming continuous preferences, $\mathrm{R}^{>i}(\mathbf{x}^i*)$ is an open set $\forall\ i$, so $\mathrm{R}^{>}(\mathbf{x}*)$ is an open set with $\boldsymbol{\omega}$ on the boundary... really don't need this assumption for the proof, but we'll use it for the intuition (plus we need continuity for Part 2 anyway)

D. $\exists$ vector $\mathbf{p}\neq\mathbf{0}$ and number r such that $\mathbf{p}\cdot\boldsymbol{\omega}=r$ (a hyperplane through the endowment point) and $\mathbf{p}\cdot\mathbf{z}>r$ $\forall$ $\mathbf{z}\in\mathrm{R}^{>}(\mathbf{x}*)$ (i.e., $\mathrm{R}^{>}(\mathbf{x}*)$ is above the hyperplane)

Proof: separating hyperplane theorem (technically the "supportive separating hyperplane theorem" because the hyperplane goes through the point)



Intuition - since $\boldsymbol{\omega}$ is on the boundary of $\mathrm{R}^{>}(\mathbf{x}^*)$, a hyperplane can't be between them, but it can go through $\boldsymbol{\omega}$ without going through $\mathrm{R}^{>}(\mathbf{x}^*)$ ... this is where we're assuming continuous preferences, but it's not vital to the proof because continuity is really worst case ($\boldsymbol{\omega}$ on boundary); otherwise, the regular separating hyperplane theorem would work because we can actually have a hyperplane between $\boldsymbol{\omega}$ and $\mathrm{R}^{>}(\mathbf{x}^*)$

E. $\mathbf{p}$ and r define a price system

<u>Proof</u>: take an allocation $\mathbf{x}$ (not aggregate consumption which is just totals of each commodity, but a vector that says how much of each commodity each consumer gets; i.e., it defines each $\mathbf{x}^i$)

Assume $\mathbf{x}^i \; \mathrm{R}^i \; \mathbf{x}^{i}*$ (so $\mathbf{x}$ is either in $\mathrm{R}^{>}(\mathbf{x}^*)$ or on the boundary)

$\therefore \; p\cdot\sum_{i=1}^{m}\mathbf{x}^i \geq r$

We know If $\mathbf{x}^i \; \mathrm{P}^i \; \mathbf{x}^{i}*$ then $p\cdot\sum_{i=1}^{m}\mathbf{x}^i > r$ by separating hyperplane theorem (from part D), but to prove ≥ from above, we'll use proof by contradiction:

Assume $p\cdot\sum_{i=1}^{m}\mathbf{x}^i < r$

Because of <u>local nonsatiation</u> $\exists \; \bar{\mathbf{x}}^i$ "near" $\mathbf{x}^i$ with $\bar{\mathbf{x}}^i \; \mathrm{P}^i \; \mathbf{x}^i$

Because of <u>transitivity</u> $\bar{\mathbf{x}}^i \; \mathrm{P}^i \; \mathbf{x}^{i}*$

Also, because we can make $\bar{\mathbf{x}}^i$ as close as we want to $\mathbf{x}^i$ (definition of local nonsatiation), we can ensure $p\cdot\sum_{i=1}^{m}\bar{\mathbf{x}}^i < r$

But because $\bar{\mathbf{x}}^i \; \mathrm{P}^i \; \mathbf{x}^{i}* \; \forall \; i$, we know $\sum_{i=1}^{m}\bar{\mathbf{x}}^i \in \mathrm{R}^{>}(\mathbf{x}^*)$ so from what we found in part D, we know $p\cdot\sum_{i=1}^{m}\bar{\mathbf{x}}^i > r$ ... that's a contradiction

Intuition - found a point "near" $\mathbf{x}$ that is affordable (because we assumed $\mathbf{x}$ is affordable); that new point is better than $\mathbf{x}$ because of local nonsatiation; that means $\mathbf{x}^*$ isn't PO because we found an affordable point that is better ∴ our assumption about $\mathbf{x}$ being affordable was wrong

Result - every point at least as good as $\mathbf{x}^*$ is on or above the hyperplane defined by $\mathbf{p}\cdot\boldsymbol{\omega} = r$

F. $p\cdot\sum_{i=1}^{m}\mathbf{x}^{i}* = r$

<u>Proof</u>: in PO allocation $\sum_{i=1}^{m}\mathbf{x}^{i}* = \sum_{i=1}^{m}\boldsymbol{\omega}^i$

We know $\mathbf{p}\cdot\boldsymbol{\omega} = r$ (part D) and $\boldsymbol{\omega} \equiv \sum_{i=1}^{m}\boldsymbol{\omega}^i$ (part C) $\therefore \mathbf{p}\cdot\sum_{i=1}^{m}\boldsymbol{\omega}^i = \mathbf{p}\cdot\sum_{i=1}^{m}\mathbf{x}^{i}* = r$

Technicality - we didn't get into this too much, but technically, local nonsatiation isn't enough to ensure then entire endowment is consumed (e.g., people living on lake don't consume all the water); we need to have either free disposal or strict monotonicity from at least 1 person for each commodity

G. If $\mathbf{x}^i \ \mathrm{R}^i \ \mathbf{x}^i{*}$, then $\mathbf{p}\cdot\mathbf{x}^i \geq \mathbf{p}\cdot\mathbf{x}^i{*}$ (i.e., $\mathbf{x}^i{*}$ is no more expensive than things that are weakly preferred to it)

<u>Proof</u>: (by contradiction) assume $\mathbf{p}\cdot\mathbf{x}^i < \mathbf{p}\cdot\mathbf{x}^i{*}$

We can form an allocation with $\mathbf{x}^i$ and $\mathbf{x}^k{*}$ (for $k \neq i$)... that is, everybody gets the same thing they did under the PO allocation $\mathbf{x}{*}$ (so they're indifferent) and consumer i gets $\mathbf{x}^i$ which is weakly preferred

Because we assumed $\mathbf{p}\cdot\mathbf{x}^i < \mathbf{p}\cdot\mathbf{x}^i{*}$, we know $\mathbf{p}\cdot\left(\mathbf{x}^i + \sum_{k\neq i}\mathbf{x}^k\right) < \mathbf{p}\cdot\sum_{i=1}^{m}\mathbf{x}^i{*} = r$ ... but that contradicts what we showed in part E

H. $I^i = \mathbf{p}\cdot\mathbf{x}^i{*} \Rightarrow I^i - \mathbf{p}\cdot\boldsymbol{\omega}^i = T^i$ (i.e., transfers exists that allow us to go from $\boldsymbol{\omega}$ to $\mathbf{x}{*}$)

<u>Proof</u>: $\sum_{i=1}^{m} T^i = \sum_{i=1}^{m} I^i - \mathbf{p}\cdot\sum_{i=1}^{m}\boldsymbol{\omega}^i = r - r = 0$

Part 2 - if preferences are continuous for each individual and if at the QE each individual has a cheaper bundle in the budget set, then the QE is a CE

Cheaper Bundle - Several assumptions will guarantee there is a cheaper bundle available (only need one of these):

Positive Endowment - if each person has a positive endowment of each good (i.e., $\boldsymbol{\omega}^i >> \mathbf{0} \ \forall \ i$); this assumption is actually too strong (and not realistic)

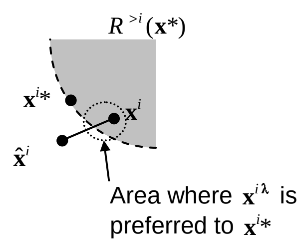
Positive Prices - if all prices are greater than zero ($\mathbf{p} >> \mathbf{0}$) and each individual has a positive endowment of at least 1 commodity; this is still too strong (it's possible to have goods with zero price)

Single Commodity - a single commodity has a positive price and positive endowment for each consumer (e.g., labor [everybody has time] and wages)... this is most realistic way to ensure cheaper bundles exist

* We'll assume $\boldsymbol{\omega}^i >> \mathbf{0}$ because it's easiest to use ("with immense amount of mathematical complication" we can weaken this assumption and get the same results)

<u>Proof</u>: assume $\mathbf{x}{*}$ is a QE that is not a CE

That means $\exists \ \mathbf{x}$ such that $\mathbf{x}^i \ \mathrm{P}^i \ \mathbf{x}^i{*}$ and $\mathbf{p}\cdot\mathbf{x}^i = I^i$ (i.e., a bundle that's better than $\mathbf{x}{*}$ but that costs the same amount... which still allows $\mathbf{x}{*}$ to be a QE [cost minimizing], but not a CE [utility maximizing])



We know $\mathbf{p}\cdot\mathbf{x}^i{*} = I^i$ because $\mathbf{x}{*}$ is a QE

Because there is a <u>cheaper bundle</u>, $\exists \ \hat{\mathbf{x}}^i$ with $\mathbf{p}\cdot\hat{\mathbf{x}}^i < I^i$

We can't have $\hat{\mathbf{x}}^i \ \mathrm{R}^i \ \mathbf{x}^i{*}$ because $\mathbf{x}{*}$ is a QE so we know $\mathbf{x}^i{*} \ \mathrm{P}^i \ \hat{\mathbf{x}}^i$

Consider $\mathbf{x}^{i\lambda} \equiv \lambda\mathbf{x}^i + (1-\lambda)\hat{\mathbf{x}}^i$

Since $\mathbf{p}\cdot\mathbf{x}^i = I^i$ and $\mathbf{p}\cdot\hat{\mathbf{x}}^i < I^i$, we know $\mathbf{p}\cdot\mathbf{x}^{i\lambda} < I^i$

From <u>continuity</u> of preferences, for λ near 1, $\mathbf{x}^{i\lambda} \ \mathrm{P}^i \ \mathbf{x}^i{*}$ (because $\mathbf{x}^i \ \mathrm{P}^i \ \mathbf{x}^i{*}$)

That means $\mathbf{x}^{i\lambda}$ is a cheaper way to get at least as much utility as $\mathbf{x}^i{*}$ so $\mathbf{x}^i{*}$ is not a QE... contradiction

Intuition - continuity lets us draw budget line tangent to both indifference curves in the Edgeworth box

Aside ($\mathbf{x}^i \geq \mathbf{0}$ vs. $\mathbf{x}^i \in X$)

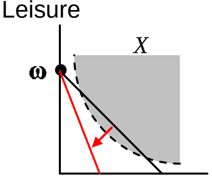
Rather than assume consumption occurs with nonnegative quantities of each commodity (i.e., $\mathbf{x}^i \geq \mathbf{0}$), Mas Colell uses a closed and convex **consumption set** (also called a **survival set**)

Why Use It - may be points where consumption is so small that individual can't survive

Sen - studied famines; said there are the usual causes of famines (drought, war, natural disaster), but (theoretically) famine could also be caused by an economic system

$\boldsymbol{\omega}^i \notin X$ - some people have endowment vector that is outside their survival set... pretty common (e.g., laborer has no food, but trades his labor for food so his budget line goes through the survival set)

Problem - if budget line is close to the boundary of the survival set, a drop in wages or increase in price of food could move the budget line below the survival set; traditionally, the price of food rises because of a shortage, but Sen argued it could also be caused by a boom in one sector of the economy (those people buy more food which drives the price up and hurts the people in the sector of the economy that didn't change)



Why We Ignore It - makes math harder

Issues with FTWE

2nd FTWE Issues

Static - in 2nd FTWE, we assumed we could set up endowments and then get a CE

Dynamic - real world is dynamic (inter-temporal) model so "endowment" for given period is based on individual decisions in prior periods;

Beginning Period - can't go to start to redistribute endowment for 2 reasons:

1. People aren't infinitely lived (so redistributing at time 0 may not have anything to do with us now)
2. Children's endowment is based on parent's decisions... not fair to kids that parents were idiots (or selfish)

Current Period - at this point would be unfair to those who save; redistribution would destroy incentives for parents to save for their kids

Random Events - another idea for "fair" outcomes: if two people make the same decisions, but get different results (so it's not a result of poor decisions), should we redistribute for a more "fair" outcome... redistribution here is effectively insurance

Information Problem - to do redistribution, government needs to know endowments, tastes, effect of redistribution on prices... <u>everybody's</u> preferences; basically need the same amount of information as a socialist planner

Calculation Issue - prior to modern computers, how would someone crunch the numbers for this model?

Dynamic Solution - rather than calculate the one best answer, government usually decides to make small changes and watch for improvement

Principal-Agent Issue - how do we get honest revelation?

Distortion Problem - if redistribution is tied to something the recipient has to do in order to get it, there is a distortion (recipient has incentive to lie or change his behavior in order to get the redistribution)

1st FTWE Issue - **Technological Progress** - is CE best? can't get technological innovations so it may be better to secure profits for innovation (i.e., allow monopolies)... this is a dynamic issue for the 1st FTWE

Existence - have great properties for competitive equilibrium, but we still need to prove such an equilibrium exists (or at least show when it will exist)

Method for Proof -

1) Convert optimizing behavior into a system of equations

2) Look for a solution... we could have multiple solutions (e.g., constant returns to scale; perfect substitutes [or any flat portion on indifference curves]);

Assumption - strictly quasiconcave utility (i.e., strictly convex preferences) will guarantee at most 1 solution... so if CE exists, it will be unique

3) Show solution is a CE

Game Theory Analogy - we had to find a best reply correspondence for each player; then find the solution to the system of best reply correspondences (used fixed point theorem); then it was "almost trivial" to show the solution was a Nash equilibrium

Assumptions - CE will exists if preferences are [1] complete, [2] transitive, [3] continuous, [4] locally nonsatiated and [5] strictly convex

Not Necessary - could still have CE if these assumptions don't hold, but these will guarantee a CE

Pure Exchange Economy - we'll look at the easy case first to prove existence; following 3 step proof outlined above

1) Solve maximization problem: $\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$ s.t. $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$, $\mathbf{x}^i \ge \mathbf{0}$

Solution gives vector of demands: $\mathbf{x}^i* = \mathbf{x}^i(\mathbf{p}) = \mathbf{D}^i(\mathbf{p},\boldsymbol{\omega}^i)$

Excess Demand - $\mathbf{z}^i(\mathbf{p},\boldsymbol{\omega}^i) \equiv \mathbf{D}^i(\mathbf{p},\boldsymbol{\omega}^i) - \boldsymbol{\omega}^i$ ... either all terms will be zero or there must be some positive and some negative (because if all positive then individual is consuming more than his income)

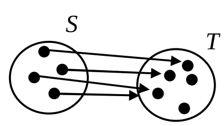
Aggregate Excess Demand - sum up over all consumers: $\mathbf{z}(\mathbf{p},\boldsymbol{\omega}) \equiv \sum_{i=1}^{m} \mathbf{z}^i(\mathbf{p},\boldsymbol{\omega}^i)$

Equilibrium - find $\mathbf{p}^*$ such that $\mathbf{z}(\mathbf{p}^*,\boldsymbol{\omega}) = \mathbf{0}$ (system of equations); if we work this out with the identities above, we have supply = demand: $\sum_{i=1}^{m} \mathbf{D}^i(\mathbf{p},\boldsymbol{\omega}^i) - \sum_{i=1}^{m} \boldsymbol{\omega}^i = \mathbf{0}$

Note1: Could technically have $\mathbf{z}(\mathbf{p}^*,\boldsymbol{\omega}) \le \mathbf{0}$ if $p_j* = 0$ for $z_j(p^*,\boldsymbol{\omega}) < 0$ (i.e., for free goods, we're allowed to have excess demand); if we assume strict monotonicity of preferences we're guaranteed to have the equality above (local nonsatiation just means consumers will always want more of one good, but strict monotonicity says they'll want more of all goods)

Note2: Since we're only talking about changing prices right now, sometimes we'll drop the endowment vector from the notation (with the assumption that it is not changing)

2) In order to prove solution exists, we need the fixed point theorem



Mapping - $S \to T$ means we assign a point in set T to every point in set S; note the definition implies each point in S gets mapped (but not necessarily to every point in T)

Goal - what to show that we can map from $P \to Z$ (i.e., from prices to excess demands) and that $\mathbf{p}^*$ maps to $\mathbf{0}$

Brouwer Fixed Point Theorem - if $f: S \to S$ (i.e., f is a mapping from set S into itself) is continuous and set S is compact (closed & bounded) and convex, then $\exists\ \mathbf{s}^* \in S$ with $f(\mathbf{s}^*) = \mathbf{s}^*$ (i.e. a point that maps to itself)

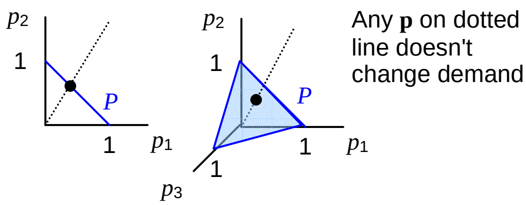
Problem - in order to get Brouwer FPT to apply, we have to use a combination of mappings... from prices to excess demands to prices

First Attempt - goal is to get $p\uparrow$ if excess demand > 0 and $p\downarrow$ if excess demand < 0; could try this: $p_j' = p_j + z_j(\mathbf{p})$

2 Problems - (1) could get $p_j' < 0$; (2) if p_j is near 0, $z_j(\mathbf{p})$ may be unbounded

Price Simplex - to get a better mapping from excess demand back into prices, we're going to transform prices to fit a price simplex (i.e., scale price vector so its magnitude is always equal to 1):

$$P = \left\{ p : p_j \geq 0 \,\&\, \sum_{i=1}^{n} p_j = 1 \right\}$$



Homogeneity of Degree Zero - property of demands: $\mathbf{D}^i(t\mathbf{p}, t\boldsymbol{\omega}) = \mathbf{D}^i(\mathbf{p}, \boldsymbol{\omega})$; that means if we find a $\mathbf{p}^*$ in P, we can scale it back into regular prices

Closed, Bounded & Convex - P satisfies all these properties

Second Attempt - using price simplex, we can now figure out a way to map from prices to prices using excess demand; there are multiple ways, but we only need one:

$$p_j' = \frac{p_j + \max[0, z_j(\mathbf{p})]}{\sum_{k=1}^{n}\left(p_k + \max[0, z_k(\mathbf{p})]\right)} = \frac{p_j + \max[0, z_j(\mathbf{p})]}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p})]}, \quad j = 1, \ldots, n$$

Check equilibrium - (recall that means $\mathbf{z}(\mathbf{p}^*, \boldsymbol{\omega}) = \mathbf{0}$) $p_j' = \dfrac{p_j + 0}{1 + 0} = p_j$ (i.e., prices don't change)

Check Still in P -

First check $p_j \geq 0$: notes that denominator is always ≥ 1; for numerator, if $z_j(\mathbf{p}) \leq 0$, we add nothing to p_j; if $z_j(\mathbf{p}) > 0$, we add a positive number to p_j; either way we end up with a numerator that is positive ∴ $p_j \geq 0$

Now check $\sum_{i=1}^{n} p_j = 1$: $\sum_{j=1}^{n} p_j' = \sum_{j=1}^{n} \left[\dfrac{p_j + \max[0, z_j(\mathbf{p})]}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p})]} \right] =$

$$\frac{\sum_{j=1}^{n}\left(p_j + \max[0, z_j(\mathbf{p})]\right)}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p})]} = \frac{1 + \sum_{j=1}^{n} \max[0, z_j(\mathbf{p})]}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p})]} = 1$$

Continuity - only remaining thing to check for Brouwer FPT to hold is that the mapping from P to P is continuous; for now, assume $z_j(\mathbf{p})$ is continuous; break down the function: (a) max of two continuous functions is continuous; (b) sum of continuous functions is continuous; ratio of continuous functions is continuous (as long as denominator is not zero... ours is ≥ 1) ∴ mapping is continuous

$\therefore$ by Brouwer FPT, $\mathbf{p}^*$ exists such that $p_j^* = \dfrac{p_j^* + \max[0, z_j(\mathbf{p}^*)]}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p}^*)]}$, $j = 1, \ldots, n$

3) Now we have to show that this is a CE; take p_j^* equation above and move denominator to LHS: $p_j^* \left[1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p}^*)]\right] = p_j^* + \max[0, z_j(\mathbf{p}^*)]$, $j = 1, \ldots, n$

p_j^* cancels: $p_j^* \sum_{k=1}^{n} \max[0, z_k(\mathbf{p}^*)] = \max[0, z_j(\mathbf{p}^*)]$, $j = 1, \ldots, n$

Multiply both sides by $z_j(\mathbf{p}^*)$: $p_j^* z_j(\mathbf{p}^*) \sum_{k=1}^{n} \max[0, z_k(\mathbf{p}^*)] = z_j(\mathbf{p}^*) \max[0, z_j(\mathbf{p}^*)]$, $j = 1, \ldots, n$

Sum over all *j*: $\left[\sum_{j=1}^{n} p_j^* z_j(\mathbf{p}^*)\right] \sum_{k=1}^{n} \max[0, z_k(\mathbf{p}^*)] = \sum_{j=1}^{j} z_j(\mathbf{p}^*) \max[0, z_j(\mathbf{p}^*)]$, $j = 1, \ldots, n$

Note: $\sum_{j=1}^{n} p_j^* z_j(\mathbf{p}^*) = 0$ ... Walras' Law (value of excess demand for all goods is zero)

$\therefore \sum_{j=1}^{j} z_j(\mathbf{p}^*) \max[0, z_j(\mathbf{p}^*)] = 0$, $j = 1, \ldots, n$

None of the terms can be negative (they're either 0 or $z_j(\mathbf{p}^*)^2 > 0$), but since they have to sum to zero, each term must be zero $\therefore$ $z_j(\mathbf{p}^*) \le 0$, $j = 1, \ldots, n$

Applying Walras' Law again, since all $p_j^* \ge 0$, we must have:

$p_j^* > 0$ with $z_j(\mathbf{p}) = 0$, or

$p_j^* = 0$ with $z_j(\mathbf{p}) < 0$ (i.e., excess supply means there's a zero price)

Note: strict monotonicity of preferences would rule out second case (there would never be a zero price), but even without that assumption we get equilibrium condition (with zero prices)

Review - we just showed that if $z_j(\mathbf{p})$ is continuous, the Brouwer Fixed Point Theorem is satisfied so a solution (i.e., competitive equilibrium) exists; now we have to fill in the blanks

Utility Maximization (review) - $\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$ s.t. $\mathbf{p} \cdot \mathbf{x}^i \le \mathbf{p} \cdot \boldsymbol{\omega}^i$, $\mathbf{x}^i \ge \mathbf{0}$

Demand - $\mathbf{x}^{i*} = \mathbf{x}^i(\mathbf{p}) = \mathbf{D}^i(\mathbf{p}, \boldsymbol{\omega}^i)$ ... solves the utility maximization problem

Adding Up Property - since preferences are strictly convex (by assumption), $\mathbf{p} \cdot \mathbf{x}^i(\mathbf{p}) = \mathbf{p} \cdot \boldsymbol{\omega}^i$

Sum across individuals: $\sum_{i=1}^{n} \mathbf{p} \cdot \mathbf{x}^i(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{p} \cdot \boldsymbol{\omega}^i$

Move terms to left side: $\sum_{i=1}^{n} \mathbf{p} \cdot (\mathbf{x}^i(\mathbf{p}) - \boldsymbol{\omega}^i) = 0$

Factor out the price vector (not dependent on *i*): $\mathbf{p}\cdot\sum_{i=1}^{n}(\mathbf{x}^i(\mathbf{p})-\boldsymbol{\omega}^i)=0$

Substitute excess demand ($\mathbf{z}^i(\mathbf{p})=\mathbf{x}^i(\mathbf{p})-\boldsymbol{\omega}^i$: $\mathbf{p}\cdot\sum_{i=1}^{n}\mathbf{z}^i(\mathbf{p})=0$

Substitute aggregate excess demand $\left(\mathbf{z}(\mathbf{p})=\sum_{i=1}^{n}\mathbf{z}^i(\mathbf{p})\right)$: $\mathbf{p}\cdot\mathbf{z}(\mathbf{p})=0$ ... Walras' Law

Walras' Law - says value of excess demand for all goods is zero; result is that excess demands are not independent functions; if we know n - 1 excess demands, we'll know the excess demand for the nth good... this doesn't really help us if there are 30 goods, but if we're only looking at 2 or 3, it definitely cuts down the workload

2 Good World - solve for general equilibrium with 1 equation and 1 unknown (price ratio)

Focus on excess demand for good 1: $z_1(p_1,p_2)=0$

Use homogeneity of degree zero in prices: $z_1(p_1,p_2)=z_1(tp_1,tp_2)=0$

Let $t=1/p_1$: $z_1(1,p_2/p_1)=0$

So now all we have to do is solve for the price ratio

(Of course, this is a little misleading saying it's only 1 equation because we have to solve three equations for each consumer to get this one equation... that is, solve the utility maximization problem for each consumer to get excess demand, then add them up to get the one equation)

Caution: the numeraire (good we choose to set price equal to 1) cannot be a good that has zero price in equilibrium... or we'd be dividing by zero

Continuity of $z_j(\mathbf{p})$ -

General Maximization Problem - $\max_{\mathbf{x}} F(\mathbf{x},\boldsymbol{\alpha})$ s.t. $\mathbf{x}\in G(\boldsymbol{\alpha})$, where $\mathbf{x}$ is a vector of decision variables and $\boldsymbol{\alpha}$ is a vector of parameters (e.g., prices and endowments); $G(\boldsymbol{\alpha})$ is the "constraint set" which identifies all possible value that $\mathbf{x}$ can take on

Maximized Value Function - $V(\boldsymbol{\alpha})\equiv\max_{\mathbf{x}} F(\mathbf{x},\boldsymbol{\alpha})$; value of the function at its maximum (e.g., indirect utility function)

Maximizer - $\mathbf{x}(\boldsymbol{\alpha})$ such that $F(\mathbf{x}(\boldsymbol{\alpha}),\boldsymbol{\alpha})\geq F(\mathbf{y},\boldsymbol{\alpha})\ \forall\ \mathbf{y}\in G(\boldsymbol{\alpha})$; the value of the decision variables that maximizes the function (e.g., demand function/correspondence)

Function - each $\boldsymbol{\alpha}$ is allowed to map to at most one value of $\mathbf{x}(\boldsymbol{\alpha})$; if function has upper hemi continuity, it is continuous

Correspondence - each $\boldsymbol{\alpha}$ is allowed to map to one or more values of $\mathbf{x}(\boldsymbol{\alpha})$; if correspondence has upper & lower hemi continuity, it is continuous

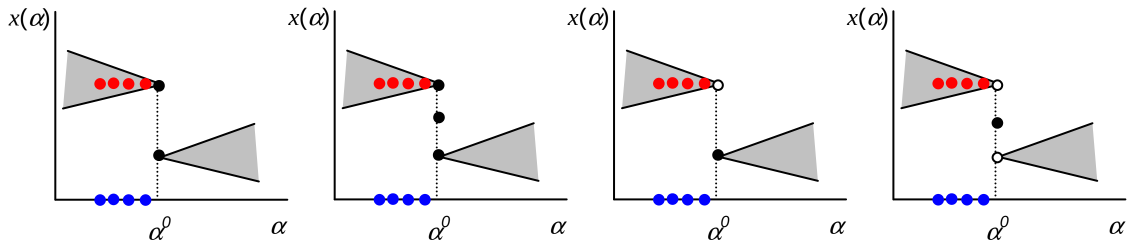
Berge Maximum Theorem - if $F(\mathbf{x},\boldsymbol{\alpha})$ is continuous in $\mathbf{x}$ and $\boldsymbol{\alpha}$ and $G(\boldsymbol{\alpha})$ is compact (closed and bounded) for each $\boldsymbol{\alpha}$ and continuous in $\boldsymbol{\alpha}$, then the maximized value function ($V(\boldsymbol{\alpha})$) is continuous and the maximizer ($\mathbf{x}(\boldsymbol{\alpha})$) is upper hemi continuous

Note1: if $G(\boldsymbol{\alpha})$ is not bounded for a given set of parameters, then the problem may not have a solution (e.g., zero price could result in infinite demand for good so there's no way to maximize utility)

Note2: If a parameter does not enter a function (as they don't in utility maximization we're studying), the function is continuous in that parameter

Upper Hemi Continuity - this is the "sort of" continuous we talked about in micro; Consider sequence of points α^n that converges to α^0 (blue dots in graphs); upper hemi continuity says that any series determined by $x(\alpha^n)$ (red dots) converges to a point in $x(\alpha^0)$

Are UH Continuous | Are Not UH Continuous



Formally - given the convergent sequence $\alpha^n \to \alpha^0$, then any sequence $y^n \in x(\alpha^n)$, with $y^n \to \bar{y}$ has $\bar{y} \in x(\alpha^0)$

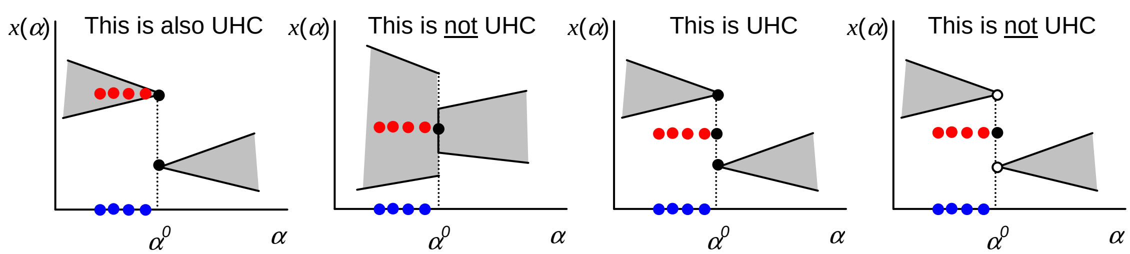
Another Way - if sequence of points in the correspondence converges to $(\alpha^0, \bar{y})$, then $(\alpha^0, \bar{y})$ must be in the correspondence

Convergence - only look at convergent sequences; some sequences will jump back and forth and the limit doesn't exist; for these sequences, we can use sub-sequence that will converge

Lower Hemi Continuity - works backwards from UHC; take any point $\bar{y}$ in the correspondence at α^0; for any sequence of points α^n that converges to α^0, there exists a sequence in the correspondence that converges to $\bar{y}$

Difference - LHC is a very subtle difference (for me anyway) from UHC; basically, UHC says we look at a sequence in the correspondence to see if it converges to a point in the correspondence; LHC says we look at a point in the correspondence and then see if we can find a sequence in the correspondence that converges to that point... clear as mud?

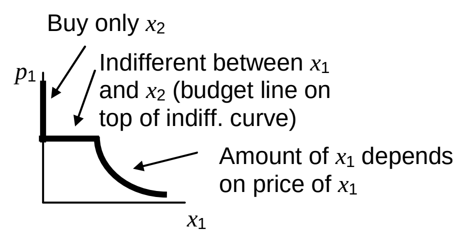
Are LH Continuous | Are Not LH Continuous



Formally - take any $\bar{y} \in x(\alpha^0)$; for any convergent sequence $\alpha^n \to \alpha^0$ $\exists$ $y^n \in x(\alpha^n)$ such that $y^n \to \bar{y}$

Strict Convexity Assumption - if preferences are strictly convex (i.e., $G(\boldsymbol{\alpha})$ is a strictly convex set; $F(\mathbf{x}, \boldsymbol{\alpha})$ is strictly quasiconcave), then there will be a unique optimizer so $\mathbf{x}(\boldsymbol{\alpha})$ is a function (not a correspondence)

Standard Correspondence - if goods are perfect substitutes, the demand correspondence is not continuous (because it's not a function); it is UHC, but not LHC (to show not LHC,



pick a point in the middle of the flat section... no sequence will converge to that point)

Consumer Problem (review again) - $\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$ s.t. $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$, $\mathbf{x}^i \ge \mathbf{0}$

To get Berge Maximum Theorem to hold, we need to show: (1) $u^i(\mathbf{x}^i)$ is continuous wrt $\mathbf{x}^i$; (2) $u^i(\mathbf{x}^i)$ is continuous wrt $\mathbf{p}$ and $\boldsymbol{\omega}$; (3) $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ is closed and bounded; (4) $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ is continuous in $\mathbf{p}$ and $\boldsymbol{\omega}$

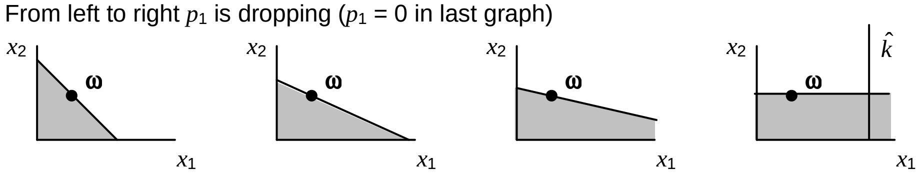
(1) $u^i(\mathbf{x}^i)$ is continuous wrt $\mathbf{x}^i$ ... by assumption

(2) $u^i(\mathbf{x}^i)$ is continuous wrt $\mathbf{p}$ and $\boldsymbol{\omega}$... automatic because $\mathbf{p}$ and $\boldsymbol{\omega}$ are not in $u^i(\mathbf{x}^i)$

(3a) $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ is closed... guaranteed by the weak part of the inequality (i.e., =)

(3b) $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ is bounded... problem if any $p_j = 0$ because the feasible region will be unbounded (can get as much x_j as you want)

From left to right p_1 is dropping (p_1 = 0 in last graph)

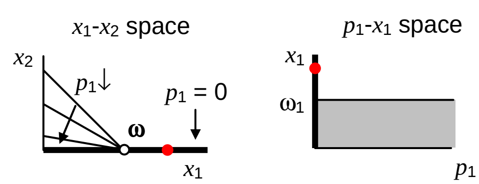


Solution - artificially create boundedness by adding constraint: $\mathbf{x}^i \le \hat{k}$; set $\hat{k}$ big enough that it can't be an equilibrium quantity (e.g., $\hat{k} > \max_j\left(\sum_i \boldsymbol{\omega}^i_j\right)$... bigger than the largest aggregate endowment)

(4) $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ is continuous in $\mathbf{p}$ and $\boldsymbol{\omega}$... need to show UHC and LHC

UHC - easy (but we didn't do it)

LHC - run into trouble with Arrow's Exceptional Case (i.e., zero price with endowment on axis... in this case feasible set is UHC but not LHC)



Red dot is feasible point on budget line at p_1 = 0; can't get sequence to converge to this point (∴ not LHC)

Solution - $\omega^i_j > 0\ \forall\ i, j$ (i.e., every individual has positive endowment of every good)... this solves the technical problem, but it's an unrealistic assumption

Other Solution - single valued demand function (i.e., strictly convex preferences) with UHC will be continuous

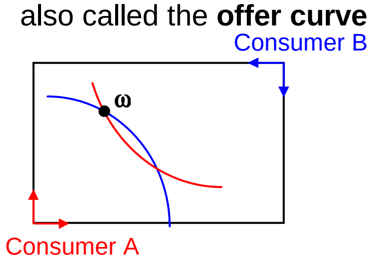
∴ Berge Maximum Theorem holds so we can say $z_j(\mathbf{p})$, that means we can use the Brouwer Fixed Point Theorem (assuming complete, continuous, transitive, locally nonsatiated, and strictly convex preferences) to guarantee that a competitive equilibrium exists

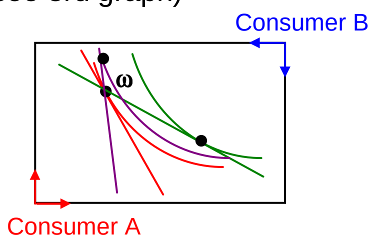
Offer Curves

Consider Edgeworth box with endowment point ω; each consumer has an indifference curve through the endowment point (see 1st graph)

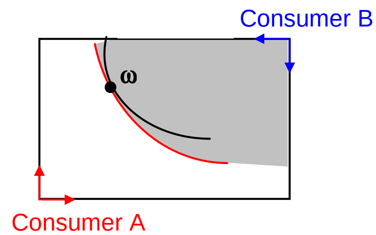
Any budget line through this endowment point will have an indifference curve at least as good as the original one that is tangent to the budget line; this determines the point that maximizes utility for the consumer (at the given price ratio) (2nd graph shows 3 budget lines for consumer A; the red is tangent to the original indifference curve)

If we find the tangency for all possible budget lines, we get a "**price consumption curve**" also called the **offer curve** (see 3rd graph)



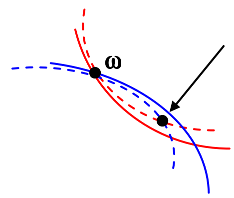


Each budget line has an indiff curve that is tangent to it



Offer curve is locus of these tangency points; it must lie above the original indiff curve

Offer Curve - locus of points that maximize utility for all possible price ratios given the initial endowment point ω; ∴ any point on the offer curve has an indifference curve for the consumer that is tangent to the budget line from ω to the point on the offer curve



Equilibrium - any point where offer curves intersect; except at ω, unless ω has indifferences curves tangent (i.e., ω is PO)

Shape - don't know anything about the shape of the offer curve; just know that it is above the indifference curve that goes through the endowment point ω and it is tangent to that indifference curve at the point ω (consumer can do no worse than this indifference curve because he can always consume his endowment)

Equilibrium - the intersection of each consumer's offer curve is a competitive equilibrium (and a PO point);

PO - draw the budget line from ω to the intersection; both consumers have indifference curves tangent to the budget line (and each other) at that point

CE - draw the budget line from ω to the intersection, both consumers maximize utility for that price ratio at that point; given amounts consumers want to trade at the price ratio is the same, the market clears (i.e., amount of good 2 that consumer A wants to give up to gain a given amount of good 1 is the same amount of good 2 that consumer B is willing to accept for the amount of good 1 he wants to get rid of... makes perfect sense!)

Multiple Equilibria - hard to draw, but there's no theory on which equilibrium point would be implemented

Problem Set 1 - lexicographic preferences aren't continuous so you can't really draw the offer curves, but can draw budget lines and look at where offer curve would be (i.e., best point for each consumer); if both have same point, the offer curves intersect so that point is a CE

Convex Preferences

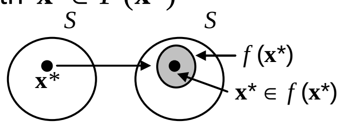
Now relax assumption about strictly convex preferences to just convex preferences... that means out demand function becomes a demand correspondence so we can't appeal to the Brouwer FPT

Correspondence - maps point into a set; $f : S \to 2^T$

Power Set - set of all subsets; for a finite set with n elements, the number of elements in the power set is equal to 2^n (which explains the notation for power set)

Example - T = {1, 2, 3}; 2^T = {{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, ∅}... $2^3 = 8$ elements

Kakutani Fixed Point Theorem - if $f : S \to 2^S$ is compact valued, convex valued, and upper hemi continuous, and S is compact and convex, then $\exists\ \mathbf{x}^* \in S$ with $\mathbf{x}^* \in F(\mathbf{x}^*)$ (This is a generalization of Brouwer FPT)



Demand Correspondence - compact (from utility constraint); UHC (from Berge theorem); convex valued (from convex preferences)

Brouwer Technique - we had $P \to z(\mathbf{p}) \to P$, but we can't use that same technique now because $\mathbf{p} \in P$ will map to set of excess demands, not a single point

Mapping - $p_j' = \dfrac{p_j + \max[0, z_j(\mathbf{p})]}{1 + \sum_{k=1}^{n} \max[0, z_k(\mathbf{p})]}$ ... if we use this mapping, we end up with $z_j(\mathbf{p})$ returning a set of p_j' so we get a region of the price simplex rather than a single point... that region is not guaranteed to be convex if there are more than two goods

Solution - map from $X \times P$; Cartesian product of $X = \{\mathbf{x} : 0 \le x_i \le K\}$ (i.e., a "big cube") and the price simplex; where $K = m\hat{k}$ (number of consumers times size of maximum aggregate endowment... i.e., a finite number to bound the problem, but big enough to ensure an interior solution)

Compact - X and P are compact sets; the Cartesian product of compact sets is a compact set

Convex - X and P are convex sets; the Cartesian product of convex sets is a convex set

Mapping - need to get from point $(\mathbf{x}, \mathbf{p})$ to $(\mathbf{x}', \mathbf{p}')$

$\mathbf{x}' = D(\mathbf{p}) = \sum_{j=1}^{m} D^j(\mathbf{p})$ ... i.e., the optimal aggregate demand correspondence given $\mathbf{p}$

$$p_j' = \frac{p_j + \max[0, x_j - \omega_j]}{1 + \sum_{k=1}^{n} \max[0, x_k - \omega_k]}$$

$x_j - \omega_j$ is an arbitrary excess demand (not based on $\mathbf{p}$)

English - take arbitrary quantities (demand correspondence) and price and get new demand correspondence and price

Need to show $\mathbf{x}' \in X$ and $\mathbf{p}' \in P$:

$D^j(\mathbf{p})$ solves $\max_{\mathbf{x}^j} u^j(\mathbf{x}^j)$ s.t. $\mathbf{p}\cdot\mathbf{x}^j \le \mathbf{p}\cdot\boldsymbol{\omega}^j$ and $0 \le \mathbf{x}^j \le \hat{k}$ (i.e., each element of $\mathbf{x}^j$ is between 0 and $\hat{k}$); recall $K = m\hat{k}$; since each $x_l^j \le \hat{k}$, we'll have $\sum_{l=1}^{m} x_l^j \le m\hat{k} = K$ (i.e., aggregate demand for good l can't exceed K)

$\therefore\ \mathbf{x}' \in X$

Follow arguments from before (p.14) to get $p_j \ge 0$ and $\sum_{i=1}^{n} p_j = 1\ \therefore\ \mathbf{p}' \in P$

$\therefore$ the mapping above maps $X \times P$ to $2^{X\times P}$

More Details - p_j' is compact, convex, and continuous (single point); $\mathbf{x}'$ is UHC...

Berge result says sum of UHC correspondences is UHC... "don't need to worry about specific details"

Kakutani FPT holds $\therefore\ \exists\ (\mathbf{x}^*, \mathbf{p}^*)$ with $\mathbf{x}^* \in D(\mathbf{p}^*)$ & $p_j^* = \dfrac{p_j^* + \max[0, x_j^* - \boldsymbol{\omega}_j]}{1 + \sum_{k=1}^{n} \max[0, x_k^* - \boldsymbol{\omega}_k]}$

Need to show that $(\mathbf{x}^*, \mathbf{p}^*)$ is a CE:

1. $\mathbf{x}^* \in D(\mathbf{p}^*)$ means these $\mathbf{x}^*$ maximize utility given $\mathbf{p}^*$
2. Manipulate p_j^* as we did before and apply Walras' Law (top of p.15)

Externalities

McKenzie - "dependent consumer preferences"

$\max_{\mathbf{x}^1} u^1(\mathbf{x}^1, \mathbf{x}^2)$ s.t. $\mathbf{p}\cdot\mathbf{x}^1 \le \mathbf{p}\cdot\boldsymbol{\omega}^1$ and $0 \le \mathbf{x}^1 \le \hat{k}$ (note consumer 1 can't control $\mathbf{x}^2$)

In this case, Welfare theorems don't hold

McKenzie showed that equilibrium exists (but it's not PO)

Assume strict convexity of preferences

Other people's choices are parameters: $D^1(\mathbf{p}, \mathbf{x}^2)$ ($\hat{k}$ and $\boldsymbol{\omega}^1$ don't matter because they don't change)

Apply Berge Theorem... $D^1(\mathbf{p}, \mathbf{x}^2)$ continuous in $\mathbf{p}$ and $\mathbf{x}^2$ (same for $D^2(\mathbf{p}, \mathbf{x}^1)$)

Solve $D^1(\mathbf{p}, \mathbf{x}^2)$ and $D^2(\mathbf{p}, \mathbf{x}^1)$ to get reduced form demands $\hat{\mathbf{x}}^1(\mathbf{p})$ and $\hat{\mathbf{x}}^2(\mathbf{p})$

Reduced form demands may not be functions and may not be "well behaved" (continuous, convex, etc.) $\therefore$ map $P \times X^1 \times X^2$:

$\mathbf{x}^{1\prime} = D^1(\mathbf{p}, \mathbf{x}^2)$

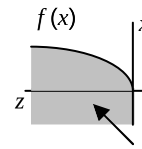
$\mathbf{x}^{2\prime} = D^2(\mathbf{p}, \mathbf{x}^1)$ ... $\mathbf{x}^1$ and $\mathbf{x}^2$ are arbitrary quantity vectors

$$p_j' = \frac{p_j + \max\left[0, \sum_{k=1}^{m} (\mathbf{x}^k - \boldsymbol{\omega}^k)\right]}{1 + \sum_{l=1}^{n} \max\left[0, \sum_{k=1}^{m} (x_l^k - \boldsymbol{\omega}_l^k)\right]}$$

This mapping will use Kakutani FPT and fixed point $(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{p}^*)$ will be CE, but not PO

Production

Simplest Case - 1 consumer, 1 firm, 1 input (z), 1 output (x), but continue to assume consumers and firm are price takers... this isn't realistic, but it's a metaphor; if you prefer, think of a million identical consumers and a million identical firms



Production Function - $f(z)$; assumed to be continuous and strictly concave; sets upper limit on output: $x \le f(z)$

Prices - price of input is w; price of output is p

Producer Problem - $\max_{x,z} \pi = px - wz$ s.t. $x \le f(z)$, $x, z \ge 0$

Input - typically consider inputs negative, but in this case, we're making it positive and accounting for the negative in the objective function (this is just a technical trick to make solving the optimization problem easier)

Rewrite It - can incorporate the constraint: $\max_{x,z} \pi = pf(z) - wz$ s.t. $z \ge 0$

2nd Order - maximizing a continuous, strictly concave objective ($f(z)$ is strictly concave and wz is linear so adding them together is strictly concave) $\therefore$ second order conditions are satisfied

Bounded - technically have same problem as before: unbounded if $w = 0$ $\therefore$ set arbitrary large upper bound Z (which won't bind when we account for consumer problem)... this is realistic because real world has resource constraints on z

Optimized Value - $\pi(p, w)$

Optimizing Value - $z^d(p, w)$ (firm's demand for input) $\Rightarrow$ $x^s(p, w) = f(z^d(p, w))$ (firm's supply of output)

Apply Berge - objective is continuous in z (by assumption) and linear in p & w (so it's also continuous in the parameters); parameters don't enter the constraint $0 \le z \le Z$ so it's also continuous in the parameters; with the bound we added, the constraint is always closed and bounded (i.e., compact) $\therefore$ Berge Maximum Theorem holds... $\pi(p, w)$ is continuous and $z^d(p, w)$ is continuous and UHC

Homogeneity - because of the structure of the problem:

$\pi(p, w)$ is homogeneous of <u>degree 1</u> in p & w...

$$\pi(tp, tw) = (tp)x - (tw)z = t(px - wz) = t\pi(p, w)$$

$z^d(p, w)$ (and hence $x^s(p, w)$)) are homogeneous of <u>degree 0</u> in p & w...

$z^d(tp, tw) = z^d(p, w)$... we'll use this later to make the input a numeraire ($w = 1$)

Consumer Utility - $u(x, z)$ is increasing in x and decreasing in z; assume monotonicity and strictly quasiconcave (i.e., strictly convex preferences)

Consumer Problem - $\max_{x,z} u(x, z)$ s.t. $px = wz + p\omega_x + \pi(p, w)$, $0 \le z \le \omega_z$

Note1: ω_z is the consumer's endowment of good z; the second constraint is what will really bind z in the producer problem so the trick we used will be OK

Note2: For the consumer, z net (i.e., endowment - consumption); think of z as labor which is really the net of time and leisure ($L = T - R$); if you view the cost of leisure as the wage (price of labor) [and ignore the other terms for now] we really have $px + wR \le wT \Rightarrow px \le wL$

Another Assumption - the consumer doesn't see that he can change $\pi(p, w)$ which seems unrealistic in the 1 consumer, 1 producer case (since he's the only consumer, he actually owns the firm), but remember, we're just using this as a metaphor... think of 1 million consumers and 1 million firms and it's more realistic

Optimized Value - $u(p,w)$ ... technically have $\boldsymbol{\omega}_x$ and $\boldsymbol{\omega}_z$ too but we're not changing them so we'll leave them out

Optimizing Value - $z^s(p,w)$ (consumer's supply of input) and $x^d(p,w)$ (consumer's demand for output)

Apply Berge - $u(x,z)$ is well behaved (by assumption); budget set is also well behaved as long as $p > 0$ (which we can either assume or argue) $\therefore$ Berge Maximum Theorem holds... $z^s(p,w)$ and $x^d(p,w)$ are continuous and UHC

Homogeneity - because of the structure of the problem:

$z^s(p,w)$ and $x^d(p,w)$ are homogeneous of <u>degree 0</u> in p & w...

$z^s(tp,tw) = z^s(p,w)$ and $x^d(tp,tw) = x^d(p,w)$

Walras' Law - monotonicity assumptions guarantees equality constraints:

$px^d(p,w) = wz^s(p,w) + p\boldsymbol{\omega}_x + \boldsymbol{\pi}(p,w)$ ... drop (p,w) for clarity

Substitute $\boldsymbol{\pi} = px^s - wz^d$: $px^d = wz^s + p\boldsymbol{\omega}_x + px^s - wz^d$

Group p and w terms: $p(x^d - x^s - \boldsymbol{\omega}_x) + w(z^d - z^s) = 0$

Excess Demands - for x: $\text{ED}^x = x^d - x^s - \boldsymbol{\omega}_x$; for z: $\text{ED}^z = z^d - z^s$

$\therefore$ $p\text{ED}^x(p,w) + w\text{ED}^z(p,w) = 0$ (Walras' Law - value of excess demands is zero)

Complication - open economy so firms owned by foreigners; consumer only gets $\alpha\boldsymbol{\pi}(p,w)$ ... makes it look like Walras' Law doesn't hold; that's why we do <u>general</u> equilibrium... have to look at all markets; if we include the foreign market, Walras' Law holds

More Complications - foreigners hold money... that means we need to treat money as a commodity and add that as a good in general equilibrium

Accounting - Walras' Law is an accounting identity... but it only works if we include all goods and all markets

Equilibrium - $z^d = z^s$ and $x^d = x^s$ (in general they're not equal); another way of looking at equilibrium is excess demands: $\text{ED}^x = 0$ and $\text{ED}^z = 0$

Apply Walras' Law - since $p\text{ED}^x + w\text{ED}^z = 0$, we know that if one of the excess demands is zero, then the other has to be zero; in general with k commodities, if we know $k-1$ excess demands are equal to zero, then the k^{th} excess demand is also zero

Apply Homogeneity - because of homogeneity of degree 0 ($\text{ED}^x(tp,tw) = \text{ED}^x(p,w)$), we only need one price: set $t = 1/w$: $\text{ED}^x(p,w) = \text{ED}^x(p/w, w/w) = \text{ED}^x(\hat{p},1)$ ... $\hat{p} = p/w$ is the price ratio

General Approach - for analyzing general equilibrium and welfare:

1. Set up the "first best" (Pareto optimal) problem... maximize joint objectives
2. Set up individual maximization problems to find competitive equilibrium (CE)
3. Determine if CE is same as PO
4. Check second order conditions... people usually forget this step

Simple Production - look at first best problem for simple production problem above:

$\max_{x,z} u(x,z)$ s.t. $x \le f(z) + \omega_z$, $x, z \ge 0$

Note: don't need a Pareto constraint to ensure other people aren't made better of because there's only 1 person in this scenario

First FTWE - solution to the first best problem is the same as the CE we found earlier

Solve PO Problem - Lagrangian: $L = u(x,z) - \lambda(x - f(z) - \omega_z)$

FOC - $\left.\begin{aligned} \frac{\partial L}{\partial x} &= \frac{\partial u}{\partial x} - \lambda = 0 \\ \frac{\partial L}{\partial z} &= \frac{\partial u}{\partial z} + \lambda f'(z) = 0 \end{aligned}\right\}$ (marginal conditions)

$\frac{\partial L}{\partial \lambda} = x - f(z) - \omega_x = 0$ (level condition)

Solve marginal conditions individually for λ and set them equal to each other:

$\lambda = \frac{\partial u}{\partial x} = -\frac{\partial u / \partial z}{f'(z)} \;\Rightarrow\; -\frac{\partial u / \partial z}{\partial u / \partial x} = f'(z) = \frac{dx}{dz}$ (this last part comes from $x = f(z)$)

Right side is slope of indifference curve: $\left.\frac{\partial x}{\partial z}\right|^{u=\text{constant}}$ $\therefore$ FOC says tradeoff is same on consumer and producer side

Solve the CE Problem - already set it up on p.22

Consumer FOC 1. $\frac{\partial u}{\partial x} - \beta p = 0$ Producer FOC 4. $\frac{\partial \pi}{\partial z} = pf'(z) - w = 0$

2. $\frac{\partial u}{\partial z} + \beta w = 0$ Market Clearing 5. $\mathrm{ED}^x(p/w) = 0$

3. $px = wz + p\omega_x + \pi$

Solve (1) and (2) for β and set them equal to each other:

$\beta = \frac{\partial u / \partial x}{p} = -\frac{\partial u / \partial z}{w} \;\Rightarrow\; -\frac{\partial u / \partial z}{\partial u / \partial x} = \frac{w}{p}$

From (4): $f'(z) = \frac{w}{p}$

Combine these two results: $-\frac{\partial u / \partial z}{\partial u / \partial x} = \frac{w}{p} = f'(z)$ ... same as PO tradeoff

From (5) and fact that $x^s = f(z)$: $\mathrm{ED}^x = 0 = x^d - x^s - \omega_x = x^d - f(z) - \omega_x$ ... which is the same as the level condition in the PO problem

2nd Order Conditions - will be the same since we assumed $f(z)$ is strictly concave and $u(x,z)$ is strictly quasiconcave

Many Producers and Consumers

Look at each **firm** separately: $\max_{\mathbf{y}^j} \boldsymbol{\pi}^j = \mathbf{p} \cdot \mathbf{y}^j$ s.t. $f^j(\mathbf{y}^j) \ge \mathbf{0} \;\; \forall \; j$

$\mathbf{y}^j$ is a netput vector (negative terms are inputs; positive terms are outputs)

Maximized value: $\boldsymbol{\pi}^j(\mathbf{p})$; maximizing value: $\mathbf{y}^j$

Look at each **consumer** separately: $\max_{\mathbf{x}^k} u^k(\mathbf{x}^k)$ s.t. $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i + \sum_j \theta^{ij}\boldsymbol{\pi}^j(\mathbf{p})$

Net seller for good k - $x_k^i - \omega_k^i < 0$

Net buyer (consumer) for good k - $x_k^i - \omega_k^i > 0$

Supply = Demand - $\sum_i x_k^i - \omega_k^i = \sum_j y_k^j$

Good News - existence and FTWE proofs are essentially the same; just more complicated math

Quote of the Day - "Purpose of this course is not advanced mathematics" - Slutsky

Mindset - general equilibrium is a mindset; recognize that there are economy-wide effects; need to account for everything; don't always need complicated model (e.g., Ricardian Technology)

Ricardian Technology - very simple model that satisfies general equilibrium (i.e., captures everything); usually use this to study consumer behavior because it's easy to work with

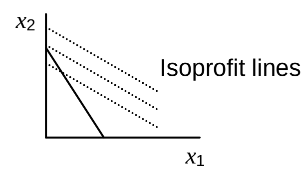
Input - single input (L for labor) with constant returns to scale ($\therefore$ technology is linear)

L_j is amount of input to produce good j $\therefore$ $x_j = \alpha_j L_j$

Technology Coefficient - α_j denotes number of units of output that can be produced by one unit of input

Input Available - $\overline{L}$

Production Possibilities Frontier - $L_j = \dfrac{x_j}{\alpha_j} \Rightarrow \overline{L} = \sum_j L_j = \sum_j \dfrac{x_j}{\alpha_j}$;



slope is linear; for 2 goods, slope depends on α_1 and α_2

Homogeneity - take advantage of this (of degree 0) to set $p_L = 1$

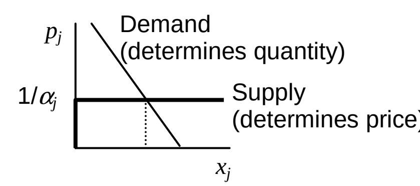
Firm - maximizing profit will always result in corner solution... unless isoprofit lines are parallel to PPF; that means ratio of prices is ratio of technology coefficients

In Math - $\max_{x_j} \boldsymbol{\pi}_j = p_j x_j - 1L_j = p_j(\alpha_j L_j) - L_j = L_j(p_j\alpha_j - 1)$

Note: if $p_j\alpha_j - 1 > 0$, profit is increased by adding more L_j; if $p_j\alpha_j - 1 < 0$, profit is increased by using less L_j $\therefore$ profit is maximized when $p_j\alpha_j - 1 = 0 \Rightarrow p_j = \dfrac{1}{\alpha_j}$

Key Result - with Ricardian Technology, prices are determined by production... demand doesn't' matter, except to require interior solution

Quantity - not determined by firms; firms are indifferent to level out output of each good because of constant returns to scale; quantity is determined by consumers:

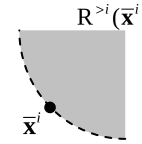


Consumer - $\max_{\mathbf{x}^i} u^i(\mathbf{x}^i)$ s.t. $\mathbf{p}\cdot\mathbf{x}^i \le \mathbf{p}\cdot\boldsymbol{\omega}^i$ (don't add profits because profits are zero); if we assume the endowment only consists of input then $\mathbf{p}\cdot\mathbf{x}^i \le p_L\overline{L}^i$

Note: $\overline{L} = \sum_i \overline{L}^i$, which is what ensures firms won't use more input that what's available

Realistic? - case where price is independent of demand (in the long run)... pizza; it's essentially the same price everywhere because of constant returns to scale

Many Producers and Consumers



PO Allocation - $\bar{\mathbf{x}}^i$ (i = 1, 2, ..., n); $\bar{\mathbf{y}}^j$ (j = 1, 2, ..., m)

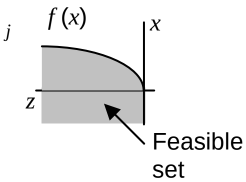
Aggregate Better Than Sets -

Individual Consumer - recall "preferred set" for individual consumer given allocation $\bar{\mathbf{x}}^i$ is

$\mathrm{R}^{>i}(\bar{\mathbf{x}}^i) \equiv \left\{\mathbf{x} : \mathbf{x}\, \mathrm{P}^i\, \bar{\mathbf{x}}^i\right\}$

All Consumers - define aggregate better than set: $\mathrm{R}^{>}(\bar{\mathbf{x}}) = \sum_{i=1}^{n} \mathrm{R}^{>i}(\bar{\mathbf{x}}^i)$ (set addition; p.8)

All Producers - define aggregate technology (i.e., feasible production) set: $Y = \sum_{j=1}^{m} Y^j$



Convexity Everywhere -

$\mathrm{R}^{>i}(\bar{\mathbf{x}}^i)$ is convex (proved on p.9)

$\mathrm{R}^{>}(\bar{\mathbf{x}})$ is convex (sum of convex sets is convex; proved on p.8)

Y^j is convex (by assumption)

Y is convex (sum of convex sets)

$Y + \{\boldsymbol{\omega}\}$ is convex (sum of convex sets)... set addition

$\boldsymbol{\omega} \in Y + \{\boldsymbol{\omega}\}$ - this follows from possibility of inaction (i.e., $\mathbf{0} \in Y$); also since **0** is on the boundary of Y, $\boldsymbol{\omega}$ is on the boundary of $Y + \{\boldsymbol{\omega}\}$

Pareto Optimal - $\mathrm{R}^{>}(\bar{\mathbf{x}}) \cap Y + \{\boldsymbol{\omega}\} = \varnothing$; intersection of better than set and feasible set is empty (i.e., there's no feasible alternative that people prefer)

Separating Hyperplane Theorem - for PO allocation $\mathrm{R}^{>}(\bar{\mathbf{x}})$ and $Y + \{\boldsymbol{\omega}\}$ are disjoint so we can use SHT; hyperplane forms price system used to get the CE that yields the same PO allocation (2nd FTWE)... rest of proof is similar to simple case (see pp.6-11)

General Pareto Problem - n consumers, m firms, K commodities

Notation - trying to be consistent; consumers & firms are superscripts and commodities are subscripts

Consumer - without loss of generality, we'll look at consumer 1, but that's just so we don't run out of letters to index things

$\max_{\mathbf{x}^1, \mathbf{x}^i, \mathbf{y}^j} u^1(\mathbf{x}^1)$ — Lagrange Multipliers

		Lagrange Multipliers
s.t.	$u^i(\mathbf{x}^i) \geq \bar{u}^i$, i = 1, 2, ..., n (Pareto constraints)	$\boldsymbol{\lambda}^i$
	$F^j(\mathbf{y}^j) \leq 0$, j = 1, 2, ..., m (Feasible production)	β^j
	$\sum_i \mathbf{x}^i \leq \sum_j \mathbf{y}^j + \boldsymbol{\omega}$ (1 constraint per commodity)	θ_k

(aggregate consumption = aggregate production + total endowment)

Variables -	$K(m+n)$	$n+1$	m	K
	decision $(\mathbf{x}^i, \mathbf{y}^j)$	$\boldsymbol{\lambda}^i$	β^j	θ_k

Standardize - to ensure multipliers are ≥ 0, we want to set this up in standard form:

$\max_{\mathbf{x}^1, \mathbf{x}^i, \mathbf{y}^j} u^1(\mathbf{x}^1)$

s.t. $\bar{u}^i - u^i(\mathbf{x}^i) \leq 0$, i = 2, ..., n (1 for each consumer other than consumer in objective)

$F^j(\mathbf{y}^j) \leq 0$, j = 1, 2, ..., m

$$\sum_i \mathbf{x}^i - \sum_j \mathbf{y}^j - \boldsymbol{\omega} \leq \mathbf{0}$$

Lagrangian - $\ell = u^1(\mathbf{x}^1) - \sum_i \boldsymbol{\lambda}^i [\bar{u}^i - u^i(\mathbf{x}^i)] - \sum_j \beta^j [F^j(\mathbf{y}^j)] - \sum_k \boldsymbol{\theta}_k \left[\sum_i x_k^i - \sum_j y_k^i - \boldsymbol{\omega}_k \right]$

Interior Solution - all u^i quasiconcave, $F^j(\mathbf{y}^j)$ convex, etc. so second order conditions hold

FOC - partials wrt Lagrange multipliers don't give many new insights (just give constraints); still need to solve them to solve the problem, but we're just looking for insights

wrt $\mathbf{x}^1$ - $\dfrac{\partial \ell}{\partial x_k^1} = \dfrac{\partial u^1(\mathbf{x}^1)}{\partial x_k^1} - \boldsymbol{\theta}_k = 0$, $k = 1, 2, ..., K$

No Externalities - x_k^1 doesn't affect $u^i(\mathbf{x}^i)$ ... so $\dfrac{\partial u^i(\mathbf{x}^i)}{\partial x_k^1} = 0$; we'll relax this later

wrt $\mathbf{x}^i$ - $\dfrac{\partial \ell}{\partial x_k^i} = \boldsymbol{\lambda}^i \dfrac{\partial u^i(\mathbf{x}^i)}{\partial x_k^i} - \boldsymbol{\theta}_k = 0$, $i = 2, ..., n$, $k = 1, 2, ..., K$

wrt $\mathbf{y}^j$ - $\dfrac{\partial \ell}{\partial y_k^j} = -\beta^j \dfrac{\partial F^j(\mathbf{x}^j)}{\partial y_k^j} + \boldsymbol{\theta}_k = 0$, $j = 1, ..., m$, $k = 1, 2, ..., K$

Solving - we're going to manipulate the various FOCs to eliminate the Lagrange multipliers and see everything is equal at the margin; we'll focus on commodities 1 and 2 without loss of generality... just trying to keep from adding more confusing indices

MRS - marginal rate of substitution; slope of indifference curve; ratio of marginal utilities

Consumer 1: $\dfrac{\partial u^1(\mathbf{x}^1)}{\partial x_1^1} = \boldsymbol{\theta}_1$ and $\dfrac{\partial u^1(\mathbf{x}^1)}{\partial x_2^1} = \boldsymbol{\theta}_2 \;\Rightarrow\; \dfrac{\partial u^1 / \partial x_1^1}{\partial u^1 / \partial x_2^1} = \dfrac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2} = \mathrm{MRS}_{1,2}^1$

Consumer i: $\boldsymbol{\lambda}^i \dfrac{\partial u^i(\mathbf{x}^i)}{\partial x_1^i} = \boldsymbol{\theta}_1$ and $\boldsymbol{\lambda}^i \dfrac{\partial u^i(\mathbf{x}^i)}{\partial x_2^i} = \boldsymbol{\theta}_2 \;\Rightarrow\; \dfrac{\partial u^i / \partial x_1^i}{\partial u^i / \partial x_2^i} = \dfrac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2} = \mathrm{MRS}_{1,2}^i$

Result - MRS for any two goods is the same for all consumers

Production -

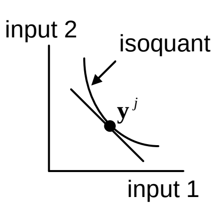
Firm 1: $\beta^1 \dfrac{\partial F^1(\mathbf{y}^1)}{\partial y_1^1} = \boldsymbol{\theta}_1$ and $\beta^1 \dfrac{\partial F^1(\mathbf{y}^1)}{\partial y_2^1} = \boldsymbol{\theta}_2 \;\Rightarrow\; \dfrac{\partial F^1 / \partial y_1^1}{\partial F^1 / \partial y_2^1} = \dfrac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2}$

Firm j: $\beta^j \dfrac{\partial F^j(\mathbf{y}^j)}{\partial y_1^j} = \boldsymbol{\theta}_1$ and $\beta^j \dfrac{\partial F^j(\mathbf{y}^j)}{\partial y_2^j} = \boldsymbol{\theta}_2 \;\Rightarrow\; \dfrac{\partial F^j / \partial y_1^j}{\partial F^j / \partial y_2^j} = \dfrac{\boldsymbol{\theta}_1}{\boldsymbol{\theta}_2}$

Result - ratio of derivatives of technology frontier wrt any two goods is the same for all firms; actually there are three different things to consider for production because with 2 goods we can have both inputs, both outputs, or 1 input and 1 output

2nd Result - MRS for any two goods for every consumer is equal to the ratio of derivatives of technology frontier... translation: slope of indifference curve is sale as slope of production possibility frontier

Note: this result actually implies the first two



2 Inputs - for inputs $\dfrac{\partial F^j / \partial y_1^j}{\partial F^j / \partial y_2^j} = \mathrm{RTS}_{1,2}^j$ = slope of isoquant (curve that shows all combinations of inputs 1 and 2 that can produce a fixed amount of output); called the marginal rate of technical substitution; sometimes abbreviated MRTS; so for any two firms we have $\mathrm{RTS}_{1,2}^{i} = \mathrm{RTS}_{1,2}^{j}$

2 Outputs - for outputs $\dfrac{\partial F^{j} / \partial y_1^{j}}{\partial F^{j} / \partial y_2^{j}} = \mathrm{MRT}_{1,2}^{j}$ = slope of PPF (curve that shows all levels of output that can be produced with fixed amount of input); called the marginal rate of transformation; so for any two firms we have $\mathrm{MRT}_{1,2}^{i} = \mathrm{MRT}_{1,2}^{j}$

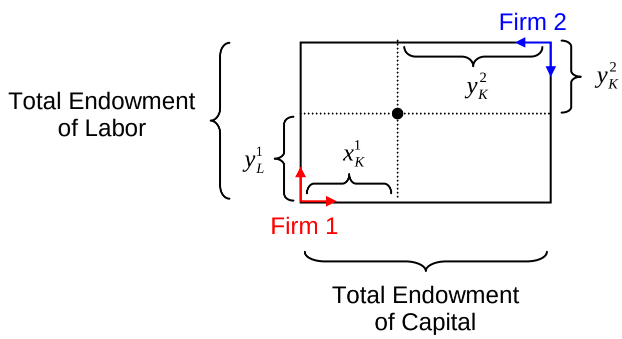
1 of Each - if good 1 is an output and good 2 is an input $\dfrac{\partial F^{j} / \partial y_1^{j}}{\partial F^{j} / \partial y_2^{j}} = \mathrm{MP}_{1,2}^{j}$ = marginal product of firm j producing output 1 with input 2; so for any two firms we have $\mathrm{MP}_{1,2}^{i} = \mathrm{MP}_{1,2}^{j}$

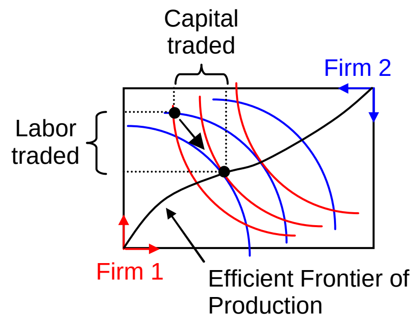
Final Result - any time two commodities are used/consumed in any 2 places (i.e. consumer or producer) in a Pareto optimal allocation, the goods trade off at the same rate... if not, there will be gains to trade so there's a Pareto improvement and the allocation wasn't PO

Example - consumer trade off between applies and leisure (marginal rate of substitution) is same as the marginal productivity of labor to produce apples

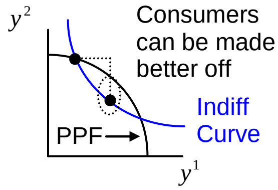
More Production - assume 2 firms with 2 inputs (capital and labor); each firm produces a single output (not the same as the other firm)

Edgeworth Box - same interpretation as for consumers, but we use isoquants instead of indifference curves; if we start at an allocation of capital and labor where two isoquants intersect, but aren't tangent there will be gains to trade; in the picture, firm 1 would trade some of its labor for some of firm 2's capital... to the point where the trade off between capital and labor is the same for both firms (isoquants are tangent)

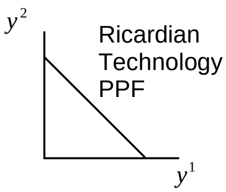




Efficient Frontier - locus of isoquant tangencies; points that are PO; can plot them in output space... yields production possibilities frontier (PPF)



PO Points - must have consumer indifference curves tangent to the PPF (if they're not there is a Pareto improvement as shown in the picture) ∴ all PO points are on the PPF



Convex - PPF is convex; strictly convex in first graph; linear in Ricardian technology

Sub Problem - since PO points must be on the PPF, we can write a sub problem to find efficient production levels (i.e., point on PPF); basically given aggregate output $\bar{y}_k$ for goods k = 2, ... K, we want to maximize the output of good 1

$$\max_{y_k} \; y_1$$

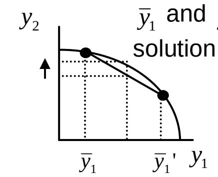
$$\text{s.t.} \quad y_k \geq \bar{y}_k, \; k = 2, \dots K$$

$$F^j(\mathbf{y}^j) \leq 0$$

$$\sum_j \mathbf{y}^j = \mathbf{y}$$

Resource Limits - consumer endowments are hard to incorporate with netput vectors; could try to put them in $\bar{y}_k$ (actually if we don't, the solution won't be feasible)

Optimized Value - $y_1(\bar{y}_2, \dots, \bar{y}_k)$ is concave



Proof: can either use definition or 2nd order conditions (not useful in this case because we don't know functional form) Pick 2 different $\bar{y}_k$ then argue convex combination gives better solution

Using PPF - general equilibrium can aggregate production to use economy-wide technology (convenient when focusing on consumers rather than firms)

Rewrite Pareto Problem -

$$\max_{\mathbf{x}^1, \mathbf{x}^i, \mathbf{y}^j} \; u^1(\mathbf{x}^1)$$

s.t. $u^i(\mathbf{x}^i) \geq \bar{u}^i$, i = 1, 2, ..., n (Pareto constraints)

$F(\mathbf{y}) \leq 0$, j = 1, 2, ..., m (Feasible aggregate production)

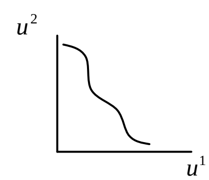
$\sum_i \mathbf{x}^i \leq \mathbf{y} + \boldsymbol{\omega}$ (1 constraint per commodity)

Finding PO Allocations - number of possible PO allocations depends on $\bar{u}^i$ (note: $\bar{u}^i$ could be set too high for available resources in which case the feasible region in the maximization problem is empty)

Optimized Value - $u^1(\bar{u}^2, \dots, \bar{u}^n)$ ... generates UPF

Utility Possibilities Frontier (UPF) - different utility levels for consumer 1 that can be achieved in PO allocation given other consumers' utility levels

Slopes Down - based on envelope theorem applied to Lagrangian:

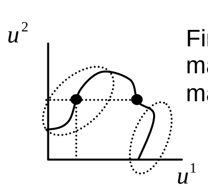


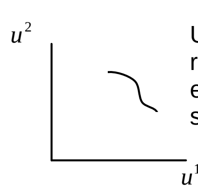
$$\ell = u^1(\mathbf{x}^1) - \sum_i \boldsymbol{\lambda}^i[\bar{u}^i - u^i(\mathbf{x}^i)] - \beta[F(\mathbf{y})] - \sum_k \boldsymbol{\theta}_k \left[\sum_i x_k^i - y_k - \boldsymbol{\omega}_k \right]$$

$\dfrac{\partial \ell}{\partial \bar{u}^i} = -\boldsymbol{\lambda}^i < 0$ ... slopes down

Concave? - not necessarily; individual utility functions are quasiconcave, but area below utility possibilities frontier may not be convex... didn't really cover why

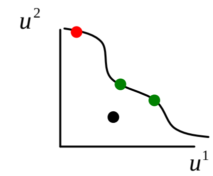
Externalities - could slope upward if there are externalities (e.g., person 1 cares about well being of person 2), but these points are not PO because both consumers can be made better off; also in this case the order of maximization matters





Social Welfare Functions

Judging Between PO Allocations - political, ethical questions; not typically a concern for economists, but economists do consider the various ways to go from an inefficient allocation to a PO point



Example - suppose we're at the inefficient point in the graph (the black one); the red point is efficient, but makes person 1 worse off... the two green points make both people better off, but not by the same amounts

Economists' Role - need method to figure out how to get to PO point and to evaluate different possible PO points

Taxi Medallion Example - large cities sold limited number of medallions required to run a taxi; the medallions ended up selling for significant amounts of money; when cities wanted to fix shortage of taxis (obvious inefficient point), issuing more medallions would enrage taxi drivers (essentially the medallions were property with value so issuing new ones would erode their value); one possible solution was to issue the new medallions to current owners of medallions

Social Welfare Function - proposed by Bergson and Samuelson; $W(u^1(\mathbf{x}^1),...,u^n(\mathbf{x}^n))$, determined by individual utility functions and overall allocation $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^n)$

Problem - u^i is not cardinal (can be transformed)... we'll get by this by assuming we <u>know</u> the correct utility function (not some transformation of it)

Form of W Matters - assume there are two people who are identical in all respects except person 2 is tone deaf (i.e. doesn't appreciate good music)

Utility - $u^1(I) > u^2(I)$ ... that is, for a given level of income person 1 will always be better off than person 2 just because he gets to enjoy something person 2 can't

Marginal Utility - $\partial u^1 / \partial I > \partial u^2 / \partial I$; this isn't necessarily implied by the scenario, but we'll assume it

Social Planner's Dilemma - you have to decide how to allocate income between the two people; 3 choices:

1) $I^1 = I^2$ ... be "fair" and give them both the same amount
2) $I^1 > I^2$ ... since person 1 has a higher marginal utility, it's "fair" to give him more income so they both value income the same on the margin
3) $I^1 < I^2$ ... since the same level of income makes person 1 better off, it's "fair" to give more income to person 2 so they have the same level of total utility

Parent Example - if social planner doesn't seem realistic, think of being a parent and trying to determine how much to leave to each child in your will

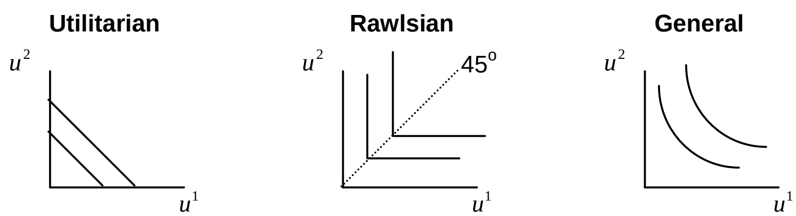
Utilitarian - treats individual utilities as <u>perfect substitutes</u> (doesn't matter who has it, just how much there is in total); this is reflected in option 2 above:

$$W_{\text{U}} = \sum_{i=1}^{n} u^i$$

Rawlsian - treats individual utilities as <u>perfect compliments</u> (only focuses on person who has the least utility); this is reflect in option 3 above:

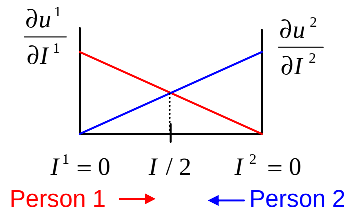
$$W_{\text{R}} = \min\left[u^1, \ldots, u^n\right]$$

Social Indifference Curves - shows various combinations of individual utility that yield same level of social welfare



Lerner - if redistributing income and know people are different but don't know which person is which, then the maximum of expected social welfare occurs where everyone has equal income

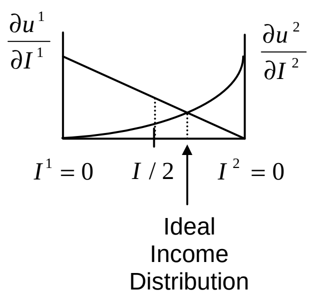
Assumption - utility is concave in income; technically not a good assumption since transformations could violate concavity, but we're assuming we know the correct utility function so we won't use any transformations

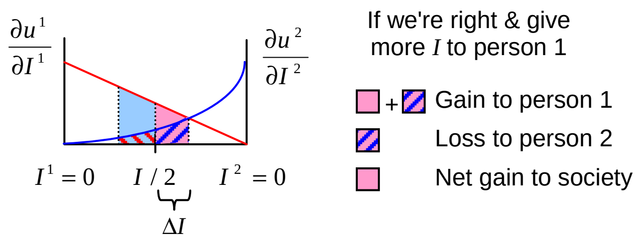


Proof with Utilitarian SWF - recall optimal distribution for Utilitarian SWF equalizes marginal utility [with respect to income] between all individuals; for simplicity, we'll assume there are only two people

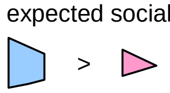
Identical People - If people are identical (have the same marginal utility), then we maximize social welfare occurs when each person has the same income

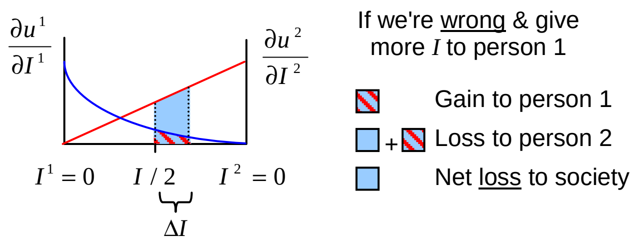
Different People - as we assumed earlier, let $\partial u^1 / \partial I > \partial u^2 / \partial I$ for all level of income; if we can identify person 1, it's easy to see the ideal income distribution gives more income to person 1; if we can't tell which person is which and we give more income to the person we think is person 1 (similarly, we could argue that we don't know if $\partial u^1 / \partial I > \partial u^2 / \partial I$ or $\partial u^2 / \partial I > \partial u^1 / \partial I$), there's a 50% chance of having a net gain to society and a 50% chance for a net loss to society and the loss exceeds the gain





Expected loss > expected gain ∴ $I/2$ maximizes expected social welfare





Proof with Rawlsian SWF - will be similar; in fact, Lerner's theorem holds for all social welfare functions when we can't identify the individuals (i.e., **absence of information**)

Real World - have to induce people to reveal information; in tone deaf example, if we want to identify those that are not tone deaf, we could easily do that (as long as self identification was beneficial); if it were not beneficial, people who were not tone deaf could easily fake it so we couldn't tell people apart

Purposes of Social Welfare Function -

Distributional Consequences - method to evaluate efficiency; just look at that

Simplifies Pareto Optimality Problem - look at this next

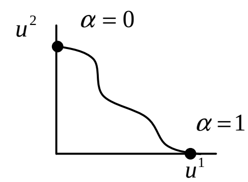
Pareto Problem - as we studied it before:

$$\max_{\mathbf{x}^i} \; u^1(\mathbf{x}^1)$$

s.t. $u^i(\mathbf{x}^i) \geq \bar{u}^i$, $i = 2, ..., n$

$F(\mathbf{x}) \leq \mathbf{0}$ ... technological feasibility (incorporates endowments)

This problem gives us the utility possibility frontier by varying $\bar{u}^i$



Alternative Formulation -

$$\max_{\mathbf{x}^i} \; \sum_{i=1}^{n} \alpha^i u^i(\mathbf{x}^i)$$ .... e.g., $\alpha u^1 + (1-\alpha)u^2$ (in graph)

s.t. $F(\mathbf{x}) \leq \mathbf{0}$ ... technological feasibility (incorporates endowments)

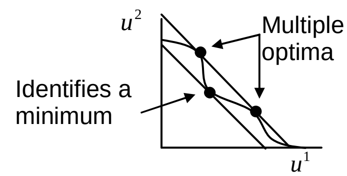
This problem gives us the utility possibility frontier by varying α^i

Alternative Constraint - rather than technological feasibility, some theorists simply write "subject to utility possibilities frontier"

Maximized Value - will be a function of $\boldsymbol{\alpha}$

Any SWF - by changing $\boldsymbol{\alpha}$, this objective function could emulate any social welfare function

Same Problem - Lagrangian for standard Pareto problem is same as for this alternative formulation, but this one is better because it has fewer constraints and it treats each individual the same (whereas the other problem gives different answers depending on whose utility is being maximized)

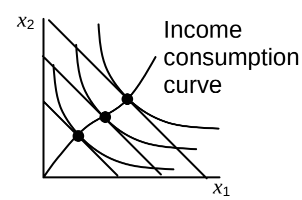


Dorfman Critique - said that running this formulation with "s.t. UPF" don't really maximize

Response - procedure not really trying to maximize utility; just trying to find points that satisfy FOC to find PO allocations; the FOC identify critical points, then we can check those points to see if they're PO

Single Individual - when can we reduce economy with many individuals to treat it as one with a single individual?... use indirect utility functions: $V(\mathbf{P},I) \equiv \text{Max } U(\mathbf{x}) \text{ st } \mathbf{P}\cdot\mathbf{x} \leq I$ and $\mathbf{x} \geq \mathbf{0}$

$\hat{W}(I^1, I^2) = W(V^1(I^1), V^2(I^2))$ ... so when can we use $\hat{W}(I^1 + I^2)$ instead?



General - we can't because marginal utility varies with income

Homothetic Preferences - if marginal utility is constant and identical for all individuals (i.e., income consumption curve is a straight line)... all this is explained in the micro handout on homothetic preferences (that we didn't read or cover in micro); identical homothetic preferences allows:

$V(\mathbf{p}, I) = f(\mathbf{p})g(I) \;\Rightarrow$ transformation $\Rightarrow \; V(\mathbf{p}, I) = f(\mathbf{p})I$

Not Valid - empirical research shows that ICC is not linear

Samuelson - identified case where we can treat everyone as a single consumer: government optimally redistributes income in lump sum transfers (still not realistic, but more so than identical homothetic preferences)

$$\max_{\mathbf{x}^1, \mathbf{x}^2} \; W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2))$$ (i.e,. max Bergson-Samuelson social welfare function)

s.t. $\mathbf{x}^1 + \mathbf{x}^2 = \mathbf{x}$ (identifies total amount available of each commodity; $\mathbf{y} + \boldsymbol{\omega}$)

Samuelson Social Welfare Function - $\hat{W}(\mathbf{x}) \equiv \max_{\mathbf{x}^1, \mathbf{x}^2} W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2))$; this is a function of aggregate consumption so it's "as if" we have a single individual; this is use frequently for trade analysis (each country treated as if it has a single consumer)

Fine Print - need some properties to be satisfied for $\hat{W}(\mathbf{x})$ to be valid

Complete - yes

Continuous - Berge Maximum Thm - if W is continuous, maximized value $\hat{W}(\mathbf{x})$ will be continuous and maximizers $\mathbf{x}^1(\mathbf{x})$ & $\mathbf{x}^2(\mathbf{x})$ will be upper hemi continuous

Monotonic - yes if $u^i(\mathbf{x}^i)$ is monotonic

Convex - "tricky question"... if W is concave in u and $u^i(\mathbf{x}^i)$ is concave in $\mathbf{x}^i$, then $\hat{W}(\mathbf{x})$ is concave in $\mathbf{x}$

Proof: want to show: $\hat{W}(\lambda\tilde{\mathbf{x}}+(1-\lambda)\overline{\mathbf{x}}) \geq \lambda\hat{W}(\tilde{\mathbf{x}})+(1-\lambda)\hat{W}(\overline{\mathbf{x}}) \;\forall\; \tilde{\mathbf{x}}\;\overline{\mathbf{x}},\; 0 \leq \lambda \leq 1$

Note: $\tilde{\mathbf{x}}$ & $\overline{\mathbf{x}}$ are any two aggregate allocations; not labeling them with numbers so we don't get confused with $\mathbf{x}^1$ & $\mathbf{x}^2$ (allocations to person 1 and 2)

Notation - $\mathbf{x}^\lambda \equiv \lambda\tilde{\mathbf{x}}+(1-\lambda)\overline{\mathbf{x}}$; $\mathbf{x}^{i\lambda} \equiv \lambda\mathbf{x}^i(\tilde{\mathbf{x}})+(1-\lambda)\mathbf{x}^i(\overline{\mathbf{x}})$ ... believe it or not this is supposed to make this simpler!

From constraint: $\mathbf{x}^1+\mathbf{x}^2=\mathbf{x}$; that means $\mathbf{x}^1(\tilde{\mathbf{x}})+\mathbf{x}^2(\tilde{\mathbf{x}})=\tilde{\mathbf{x}}$ and

$\mathbf{x}^1(\overline{\mathbf{x}})+\mathbf{x}^2(\overline{\mathbf{x}})=\overline{\mathbf{x}} \;\therefore$

$$\mathbf{x}^{1\lambda}+\mathbf{x}^{2\lambda}=\left[\lambda\mathbf{x}^1(\tilde{\mathbf{x}})+(1-\lambda)\mathbf{x}^1(\overline{\mathbf{x}})\right]+\left[\lambda\mathbf{x}^2(\tilde{\mathbf{x}})+(1-\lambda)\mathbf{x}^2(\overline{\mathbf{x}})\right] =$$
$$\lambda\left(\mathbf{x}^1(\tilde{\mathbf{x}})+\mathbf{x}^2(\tilde{\mathbf{x}})\right)+(1-\lambda)\left(\mathbf{x}^1(\overline{\mathbf{x}})+\mathbf{x}^2(\overline{\mathbf{x}})\right)=\lambda\tilde{\mathbf{x}}+(1-\lambda)\overline{\mathbf{x}}=\mathbf{x}^\lambda$$

$\hat{W}(\mathbf{x}^\lambda)$ is maximum of $W(u^1,u^2)$ when $\mathbf{x}^\lambda$ is available

When $\mathbf{x}^\lambda$ is available, $\mathbf{x}^{1\lambda}$ and $\mathbf{x}^{2\lambda}$ are feasible, although may not be optimal (we just showed $\mathbf{x}^{1\lambda}+\mathbf{x}^{2\lambda}=\mathbf{x}^\lambda$ above)

That means $\hat{W}(\mathbf{x}^\lambda) \geq W(u^1(\mathbf{x}^{1\lambda}),u^2(\mathbf{x}^{2\lambda}))$

Because $u^i(\mathbf{x}^i)$ is concave in $\mathbf{x}^i$:

$$u^1(\mathbf{x}^{1\lambda})=u^1\left(\lambda\mathbf{x}^1(\tilde{\mathbf{x}})+(1-\lambda)\mathbf{x}^1(\overline{\mathbf{x}})\right) \geq \lambda u^1(\mathbf{x}^1(\tilde{\mathbf{x}}))+(1-\lambda)u^1(\mathbf{x}^1(\overline{\mathbf{x}}))$$
$$u^2(\mathbf{x}^{2\lambda})=u^2\left(\lambda\mathbf{x}^2(\tilde{\mathbf{x}})+(1-\lambda)\mathbf{x}^2(\overline{\mathbf{x}})\right) \geq \lambda u^2(\mathbf{x}^2(\tilde{\mathbf{x}}))+(1-\lambda)u^2(\mathbf{x}^2(\overline{\mathbf{x}}))$$

Because $W(u^1,u^2)$ is increasing in u:

$$W(u^1(\mathbf{x}^{1\lambda}),u^2(\mathbf{x}^{2\lambda})) \geq$$
$$W(\lambda u^1(\mathbf{x}^1(\tilde{\mathbf{x}}))+(1-\lambda)u^1(\mathbf{x}^1(\overline{\mathbf{x}})),\lambda u^2(\mathbf{x}^2(\tilde{\mathbf{x}}))+(1-\lambda)u^2(\mathbf{x}^2(\overline{\mathbf{x}})))$$

Because $W(u^1,u^2)$ is concave:

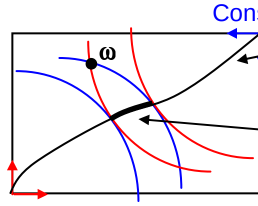
$$W(\lambda u^1(\mathbf{x}^1(\tilde{\mathbf{x}}))+(1-\lambda)u^1(\mathbf{x}^1(\overline{\mathbf{x}})),\lambda u^2(\mathbf{x}^2(\tilde{\mathbf{x}}))+(1-\lambda)u^2(\mathbf{x}^2(\overline{\mathbf{x}}))) \geq$$
$$\lambda W\left(u^1(\mathbf{x}^1(\tilde{\mathbf{x}})),u^2(\mathbf{x}^2(\tilde{\mathbf{x}}))\right)+(1-\lambda)W\left(u^1(\mathbf{x}^1(\overline{\mathbf{x}})),u^2(\mathbf{x}^2(\overline{\mathbf{x}})))\right)=$$
$$\lambda\hat{W}(\tilde{\mathbf{x}})+(1-\lambda)\hat{W}(\overline{\mathbf{x}})$$ (because we used the maximizers)

If we follow the trial of inequalities, we have

$\hat{W}(\lambda\tilde{\mathbf{x}}+(1-\lambda)\overline{\mathbf{x}}) \geq \lambda\hat{W}(\tilde{\mathbf{x}})+(1-\lambda)\hat{W}(\overline{\mathbf{x}}) \;\therefore\; \hat{W}(\mathbf{x})$ is concave in $\mathbf{x}$

Alternative - if W is quasiconcave in u and $u^i(\mathbf{x}^i)$ is concave in $\mathbf{x}^i$, then $\hat{W}(\mathbf{x})$ is quasiconcave in $\mathbf{x}$ (makes a weaker assumption so it yields a weaker result)

Core and Competitive Equilibrium



Core - most basic solution to cooperative game theory; collection of allocations that aren't blocked (see definition of blocking below)
CE in Core - actual agreement is in the core; exact location depends on bargaining skill
Edgeworth Conjecture - as you add more and more individuals, core region would shrink because people lose bargaining power; eventually shrinks to CE
Debru and Scarp - proved this for increasing the number of identical individuals
Almond - prove this for a continuum of individuals
Significance - justification of price taking assumption for large economies

"Just let me do one more thing" -Slutsky

Pure Exchange Economy - we'll study core in this scenario (easier math)
Not Institution Free - need to know what people have ownership of (i.e., individual endowments) in order to identify the core
Notation - I individuals; K commodities; $\boldsymbol{\omega}^i$ endowment vector for individual i; $\mathbf{x}$ allocation which assigns x_k^i of commodity k to consumer i (can view $\mathbf{x}$ as a $K \times I$ matrix or a $(KI) \times 1$ vector)
Coalition - subset of individuals
Blocking - coalition S can block allocation $\mathbf{x}$ if there exists an alternative allocation $\mathbf{y}$ that is a Pareto improvement for the coalition... i.e., has two properties:

1. $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \boldsymbol{\omega}^i$ (i.e., $\mathbf{y}$ is feasible for the coalition)
2. $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i) \ \forall \ i \in S$ and is strictly > for at least 1 person (i.e., coalition prefers $\mathbf{y}$)

Theorem 1 - if preferences are <u>continuous</u> and <u>strictly monotonic</u>, then every CE is in the core
Generalized 1st FTWE - since S can be everybody, every allocation in the core is PO
<u>Proof</u>: same as 1st FTWE, but do it for coalitions

Assume $(\mathbf{p}^*, \mathbf{x}^*)$ is CE but not in the core

$\therefore \exists$ a coalition S and allocation $\mathbf{y}$ with $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \boldsymbol{\omega}^i$ and $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$ (strict for at least one person in the coalition)

Since $(\mathbf{p}^*, \mathbf{x}^*)$ is CE, revealed preference and the second condition above imply

$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^{i*} \ \forall \ i \in S$ and strictly > for at least 1 person

Since it's strictly > for at least one person, the sum will be >: $\sum_{i \in S} \mathbf{p}^* \cdot \mathbf{y}^i > \sum_{i \in S} \mathbf{p}^* \cdot \mathbf{x}^{i*}$

Since $(\mathbf{p}^*, \mathbf{x}^*)$ is CE, $\sum_{i \in S} \mathbf{p}^* \cdot \mathbf{x}^{i*} = \sum_{i \in S} \mathbf{p}^* \cdot \boldsymbol{\omega}^i$

That implies $\sum_{i \in S} \mathbf{y}^i > \sum_{i \in S} \boldsymbol{\omega}^i$ which violates the first condition above

$\therefore$ the CE must be in the core

More Notation -

r replicants - # of each type of consumer; everybody of same type has same utility function $u^i(\cdot)$ and endowment $\boldsymbol{\omega}^i = \boldsymbol{\omega}^{ij}\ \forall\ j = 1, ..., r$

$\mathbf{x}^{ij}$ - allocation for j^{th} individual of type i ($i = 1,..., I$; $j = 1, ..., r$)

E^r replicated economy

c^r core in economy E^r

Equal Treatment Result - if preferences are strictly convex (i.e., strictly quasiconcave utility), no matter what r is, all individuals of the same type will consume the same bundle in equilibrium

Implication - simplifies analysis; only need to worry about I bundles (1 for each individual) rather than total number of bundles $\therefore$ solution is not dependent on total size of the economy; if an allocation is a CE for $r = 1$, it will be a CE for all r

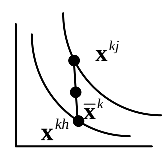
<u>Proof</u>: (by contradiction)

Assume an economy E^r with $r > 1$, has 2 individuals of the same type get different allocation in equilibrium

That means $\exists$ type k with two individuals j and h such that $\mathbf{x}^{kj} \neq \mathbf{x}^{kh}$

Without loss of generality, label j and h such that $u^k(\mathbf{x}^{kj}) \geq u^k(\mathbf{x}^{kh})$

Define $\bar{\mathbf{x}}^k = \frac{1}{r}\sum_{j=1}^{r}\mathbf{x}^{kj}$ (i.e., aggregate allocation for all type k individuals)



By strict convexity of preferences, $u^k(\bar{\mathbf{x}}^k) > u^k(\mathbf{x}^{kh})$ (i.e., the individual of type k that has the "worse" bundle would rather have the average)

Do the same for other types:

Define the average bundle for type i: $\bar{\mathbf{x}}^i = \frac{1}{r}\sum_{j=1}^{r}\mathbf{x}^{ij}$

Define the "worst" bundle for type i: $\mathbf{x}^{ih}$ (given to individual h for consistency)

Since the possibility exists that all individuals of a type (other than k) have the same bundle, the relationship between utilities is not a strict inequality: $u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^{ih})$ (in fact, this will be = if all individuals of type i have the same allocation)

Now create a coalition S with one individual of each type that gets the worst bundle for that type (i.e., individual h from each type)

Give each member in S the average bundle for his type $\bar{\mathbf{x}}^i$ instead of $\mathbf{x}^{ih}$

By construction we know part 2 of the blocking definition holds because $u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^{ih})$ $\forall\ i$ and is strictly > for the individual of type k

Now need to show the first part: that this is a feasible allocation for the coalition

From original allocations we know: $\sum_{i=1}^{I}\sum_{j=1}^{r}\mathbf{x}^{ij} = \sum_{i=1}^{I}\sum_{j=1}^{r}\boldsymbol{\omega}^{ij} = \sum_{i=1}^{I} r\boldsymbol{\omega}^i$

Divide both sides by r: $\sum_{i=1}^{I}\frac{1}{r}\sum_{j=1}^{r}\mathbf{x}^{ij} = \sum_{i=1}^{I}\bar{\mathbf{x}}^i = \sum_{i=1}^{I}\boldsymbol{\omega}^i$ ... $\therefore$ the new allocation is feasible

Since the new allocation is feasible and preferred by the coalition, they will block the original allocation; since that allocation is blocked, it's not in the core, which means it can't be a competitive equilibrium

$\therefore$ in a CE, all individuals of the same type will consume the same bundle

Core Doesn't Get Bigger - $c^1 \supset c^2 \supset \cdots \supset c^r$ (i.e., c^1 contains c^2 contains... c^{r-1} contains c^r; this can also be written with subsets: $c^r \subset c^{r-1} \subset \cdots \subset c^2 \subset c^1$

Note: these are not proper subsets because we're arguing the core doesn't get bigger; we'll argue later that it actually gets smaller as r increases, but that's a separate proof

Proof: (by contradiction)

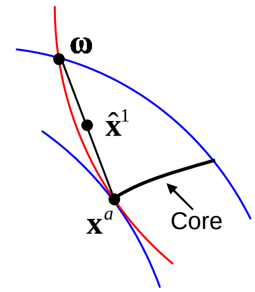
Assume $\mathbf{x} \in c^r$, but $\mathbf{x} \notin c^{r-1}$ (i.e., $c^r \not\subset c^{r-1}$)

If $\mathbf{x}$ is not in c^{r-1}, then it is blocked by some coalition S in E^{r-1}

But that coalition also exists in E^r and would also block $\mathbf{x}$ in that economy so $\mathbf{x} \notin c^r$

$\therefore$ the core for higher order replicant economies cannot contain allocations that are not already contained in the cores of the smaller economies

Core Gets Smaller -



Consider an economy with one of each type of consumer (E^1)

Define $\mathbf{x}^a$ as the allocation on the boundary of the core for given endowment $\boldsymbol{\omega}$

At $\mathbf{x}^a$ the type 1 individual is indifferent to his initial endowment: $u^1(\mathbf{x}^{a1}) = u^1(\boldsymbol{\omega}^1)$

(i.e., $\mathbf{x}^a$ on the same indifference curve as $\boldsymbol{\omega}$; see picture)

In this economy $\mathbf{x}^a$ is not blocked (because it's in the core)

Shrinking Core Theorem - If we replicate (E^2), $\mathbf{x}^a$ will no longer be in the core

Proof: Define $\hat{\mathbf{x}}^1 = \frac{1}{2}\mathbf{x}^{a1} + \frac{1}{2}\boldsymbol{\omega}^1$

Because of strictly convex preferences, we know $u^1(\hat{\mathbf{x}}^1) > u^1(\mathbf{x}^{a1})$ (see picture)

Form a coalition S made up for both type 1 individuals and one of the type 2 individuals

Give $\hat{\mathbf{x}}^1$ to both type 1's and $\mathbf{x}^{a2}$ to the type 2 guy (same as he got before)

That means the coalition would prefer this allocation to $\mathbf{x}^a$ (2 are strictly better off and 1 is at least as well off) $\therefore$ if it's feasible for them, they will block $\mathbf{x}^a$

Aggregate consumption for the coalition:

$2\hat{\mathbf{x}}^1 + \mathbf{x}^{a2} = 2(\frac{1}{2}\mathbf{x}^{a1} + \frac{1}{2}\boldsymbol{\omega}^1) + \mathbf{x}^{a2} = \mathbf{x}^{a1} + \boldsymbol{\omega}^1 + \mathbf{x}^{a2} = 2\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$ which is feasible

Note: $\mathbf{x}^{a1} + \mathbf{x}^{a2} = \mathbf{x}^a = \boldsymbol{\omega} = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$ (original allocation is feasible)

$\therefore$ the coalition will block $\mathbf{x}^a$ in E^2

This theorem generalizes to higher replications (i.e., boundary of core in E^r is blocked in E^{r+1}); the "easy" (i.e., more understandable) proof for this is in Jehle & Reny

Theorem 2 (Edgeworth Conjecture) - if $\mathbf{x} \in c^r \ \forall\ r$, then $\mathbf{x}$ is CE

Proof: (by contradiction)

Suppose $\mathbf{y} \in c^r \ \forall\ r$, but is not a CE

Prices are determined by the budget line through $\boldsymbol{\omega}$ and $\mathbf{y}$

Since $\mathbf{y} \in c^r$ (i.e., in core), consumers' indifference curves are tangent (i.e., consumers are maximizing utility), but market clears only if indifference curves are also tangent to the budget line

$\therefore$ since we're assuming $\mathbf{y}$ is not a CE, there must exist a point $\hat{\mathbf{x}}$ that is better than $\mathbf{y}$ (for one type of consumer) and is in the direction of the endowment; for now, let's assume it's better for type 1: $u^1(\hat{\mathbf{x}}^1) > u^1(\mathbf{y})$

(Argument could easily be repeated if it's type 2; see pictures below to see the difference)

By continuity of preferences, we can write $\hat{\mathbf{x}}^1 = (1-\alpha)\mathbf{y} + \alpha\boldsymbol{\omega}$ for α sufficiently small

Let $\alpha = 1/r$ for some sufficiently large, integer r

Now consider economy E^r

By assumption we should have $\mathbf{y} \in c^r$, but we'll see that a coalition will block $\mathbf{y}$

Form coalition S of all r individuals of type 1 and $r-1$ individuals of type 2

Given each type 1 individual $\hat{\mathbf{x}}^1$ and each type 2 individual $\mathbf{y}^2$

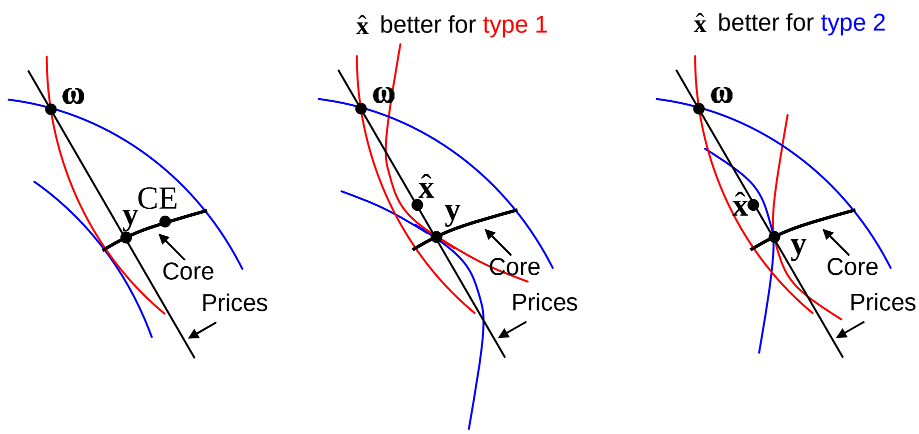
By construction, $u^1(\hat{\mathbf{x}}^1) > u^1(\mathbf{y}^1)$ and $u^2(\mathbf{y}^2) = u^2(\mathbf{y}^2)$ ... that is, some members of the coalition are strictly better off and others are at least as well off as the original allocation $\mathbf{y}$

Look at aggregate consumption for the coalition:

$$r\hat{\mathbf{x}}^1 + (r-1)\mathbf{y}^2 = r(\tfrac{1}{r}\boldsymbol{\omega}^1 + (1-\tfrac{1}{r})\mathbf{y}^1) + (r-1)\mathbf{y}^2 = \boldsymbol{\omega}^1 + (r-1)\mathbf{y}^1 + (r-1)\mathbf{y}^2 = r\boldsymbol{\omega}^1 + (r-1)\boldsymbol{\omega}^2$$

Note: $\mathbf{y}^1 + \mathbf{y}^2 = \mathbf{y} = \boldsymbol{\omega} = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$

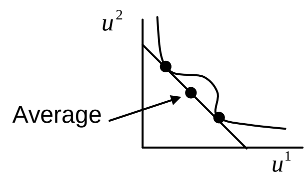
$\therefore$ this coalition will block $\mathbf{y}$ so it can't be in the core



Externalities and Public Goods

Welfare Theorems - basic assumptions we had to make:

1. **Price Takers** - core argument shows this assumption is valid for large economies; people can't lie ("misrepresent their preferences") to get out of the core
2. **Convexity** - given single individual with non-convex preferences, he could use a mixed strategy, but it doesn't make sense to talk about his average consumption; given a million of this same type of individual with half choosing one point and the other half choosing the other, and we can talk about the average consumption ∴ for large economies, the convexity assumption is valid (i.e., a large economy "convexifies" individual non-convexities)



3. **No Externalities** - decisions that affect others' welfare directly (not pecuniary [through prices])

Externalities

1st FTWE - fails if person 1 doesn't take other people's benefits/costs into account; this leads to inefficiency

Inducing Efficiency - try to get people to internalize extra benefits/costs

Pigovian Approach - government steps in with corrective subsidies/taxes; efficiency is restored <u>if</u> the subsidy/tax is the correct amount

Note: assumes lump sum taxes so we can ignore distortions from taxes

Coasian Approach - Chicago approach; government doesn't have to intervene; people can negotiate on their own

Note: assumes zero transaction/bargaining costs

Samuelson's Paper - "The Pure Theory of Public Expenditure"; derived first order conditions for Pareto optimality in presence of public goods

Definitions -

Non-Excludable - can't keep somebody from getting the good once it's supplied

Non-Rival - one person's consumption doesn't interfere with another persons; marginal cost is zero

Jointness of Supply - if the good is supplied, it's available to everyone even if excludable (e.g., non-congested bridge)

Inefficiency - good that is non-rival, but excludable can be supplied privately, but may be inefficient because MC = 0 (i.e., if firm charges price > 0, people will consume less than the socially optimal level)

Pure Public Good - is both non-rival and non-excludable

Notation -

$\mathbf{x}^i$ - vector of private good consumption for person i

$\sum_{i=1}^{n} \mathbf{x}^i$ - aggregate consumption of private goods

$\mathbf{y}$ - supply of private goods

$\mathbf{z}^i$ - vector of public good consumption for person i

$\mathbf{z}$ - supply of public goods

Pareto Optimality -

$$\max_{\mathbf{x}^i,\mathbf{z}^i} W\left(u^1(\mathbf{x}^1,\mathbf{z}^1),\ldots,u^n(\mathbf{x}^n,\mathbf{z}^n)\right)$$

s.t. $\sum_{i=1}^{n}\mathbf{x}^i = \mathbf{y}$ (market clearing for private goods; demand = supply)

$\mathbf{z}^i = \mathbf{z} \ \ \forall\, i$ (market clearing for public goods; everyone consumes same amount)

$F(\mathbf{y},\mathbf{z}) = 0$ (aggregate technology; production of private and public goods is feasible)

Note: think of objective as weighted sum of u^i's to generate the utility possibilities frontier (i.e., we're not limiting it to a specific definition of social efficiency like utilitarian or Rawlsian)

Assumptions -

1) only consider interior solution; there are some interesting results with non-negativity constraints, but we don't have time
2) at least 1 private good and 1 public good (makes the math easier)

Simplify - the formulation above is the general case which allows selective excludability so we showed how much public good each consume gets... technically could use ≤ if some are excluded; if we assume there is no one excluded, we can simplify the formulation:

$$\max_{\mathbf{x}^i,\mathbf{z}^i} W\left(u^1(\mathbf{x}^1,\mathbf{z}),\ldots,u^n(\mathbf{x}^n,\mathbf{z})\right)$$

s.t. $F\left(\sum_{i=1}^{n}\mathbf{x}^i,\mathbf{z}\right) = 0$

Lagrangian - $\ell = W\left(u^1(\mathbf{x}^1,\mathbf{z}),\ldots,u^n(\mathbf{x}^n,\mathbf{z})\right) - \lambda F\left(\sum_{i=1}^{n}\mathbf{x}^i,\mathbf{z}\right)$

FOCs -

k^{th} private good consumed by j^{th} person:

$$\frac{\partial \ell}{\partial x_k^j} = \frac{\partial W}{\partial u^j}\frac{\partial u^j}{\partial x_k^j} - \lambda\frac{\partial F}{\partial x_k^j} = 0,\ \forall\ j = 1,\ \ldots,\ n \text{ and } k = 1,\ \ldots,\ K$$

h^{th} public good consumed by any person (they all consume the same amount)

$$\frac{\partial \ell}{\partial z_h} = \sum_{j=1}^{n}\frac{\partial W}{\partial u^j}\frac{\partial u^j}{\partial z_h^j} - \lambda\frac{\partial F}{\partial z_h} = 0,\ \forall\ h = 1,\ \ldots,\ m$$

Two Private Goods - k & l for same individual i

$$\frac{\frac{\partial W}{\partial u^i}\frac{\partial u^i}{\partial x_k^i}}{\frac{\partial W}{\partial u^i}\frac{\partial u^i}{\partial x_l^i}} = \frac{\lambda\frac{\partial F}{\partial x_k^i}}{\lambda\frac{\partial F}{\partial x_l^i}} \ \Rightarrow\ \frac{\partial u^i/\partial x_k^i}{\partial u^i/\partial x_l^i} = \frac{\partial F/\partial x_k^i}{\partial F/\partial x_l^i} \ \Rightarrow\ \text{MRS}_{k,l} = \text{MRT}_{k,l}$$

Marginal rate of substitution between goods k & l = marginal rate of transformation between those two goods

Note: since MRS for any consumer = MRT (which is always the same), MRS is the same for all consumers; this is the same result as before (see p.4 of "Simple Models" notes)

Public and Private - public good h and private good 1

Take FOC of public good h (shown above): $\sum_{j=1}^{n}\frac{\partial W}{\partial u^j}\frac{\partial u^j}{\partial z_h^j} = \lambda\frac{\partial F}{\partial z_h}$

Divide both sides by $\lambda \frac{\partial F}{\partial x_1}$: $\frac{\sum_{j=1}^{n} \frac{\partial W}{\partial u^j} \frac{\partial u^j}{\partial z_h^j}}{\lambda \frac{\partial F}{\partial x_1}} = \frac{\lambda \frac{\partial F}{\partial z_h}}{\lambda \frac{\partial F}{\partial x_1}}$

$\partial F / \partial x_1$ is not dependent on the consumer, so it can be put inside the summation:

$$\sum_{j=1}^{n} \left(\frac{\frac{\partial W}{\partial u^j} \frac{\partial u^j}{\partial z_h^j}}{\lambda \frac{\partial F}{\partial x_1}} \right) = \frac{\partial F / \partial z_h}{\partial F / \partial x_1} = \text{MRT}_{z_h, x_1}$$

Now use FOC for private good 1 and consumer j: $\lambda \frac{\partial F}{\partial x_1^j} = \frac{\partial W}{\partial u^j} \frac{\partial u^j}{\partial x_1^j}$

$$\sum_{j=1}^{n} \left(\frac{\frac{\partial W}{\partial u^j} \frac{\partial u^j}{\partial z_h^j}}{\frac{\partial W}{\partial u^j} \frac{\partial u^j}{\partial x_1^j}} \right) = \sum_{j=1}^{n} \frac{\partial u^j / \partial z_h^j}{\partial u^j / \partial x_1^j} = \sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} = \text{MRT}_{z_h, x_1}$$

Samuelson Condition for Public Goods - $\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} = \text{MRT}_{z_h, x_1}$ ... sum of MRS between public and private goods for all consumers is equal to the MRT between those goods

Caution - sometimes theorists are casual about how they specify MRS

$$\text{MRS}_{ij} = \left. \frac{-dx_j}{dx_i} \right|^{u=\text{constant}} \quad \text{or} \quad \left. \frac{-dx_i}{dx_j} \right|^{u=\text{constant}}$$

But with public and private good, it matters: $\text{MRS}_{z_h, x_1^j} = \left. \frac{-dx_1^j}{dz_h} \right|^{u=\text{constant}}$ which is equal to marginal willingness to pay for the public good (i.e., holding person *j*'s utility constant, how much x_1 does person *j* have to give up for more public good z_h ∴

$\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} =$ total **social willingness to pay** for public good

Over Supply - $\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} < \text{MRT}_{z_h, x_1}$ (i.e., marginal value is less than marginal cost)

Under Supply - $\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} > \text{MRT}_{z_h, x_1}$ (i.e., marginal value exceeds marginal cost)

Duality - between public and private goods:

Quantity Conditions - $\sum_{i=1}^{n} \mathbf{x}^i = \mathbf{y}$ (private) and $\mathbf{z}^i = \mathbf{z}$ (public)

PO Conditions - $\text{MRS}_{k,l} = \text{MRT}_{k,l}$ (private) and $\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} = \text{MRT}_{z_h, x_1}$ (public)

View MRS as price ratio so $\text{MRS}_{k,l} = \text{MRT}_{k,l}$ for private goods basically says "all individuals pay the same price"; for public goods: $\sum_{j=1}^{n} \text{MRS}_{z_h, x_1^j} = \text{MRT}_{z_h, x_1}$ says that price is equal to the sum of what everybody pays

Result - "private good prices are like public good quantities" and "public good prices are like private good quantities" (mathematically)

Free Riders - since public goods are non-excludable, consumers have incentive to free-ride (not pay their "fair share" and still consume the good); for private goods, the free rider problems results in efficiency because collusion breaks down, but is the source of inefficiency for public goods

Inefficiency - because of free rider problem "voluntary approach" (asking consumers to pay for public goods by paying for the value they receive) doesn't work; usual solution is compulsory **benefit taxation**, but even that doesn't work because government can't determine MRS for each individual (because of free rider problem); that means government turns to **income taxation** assuming income is perfectly correlated with preferences for public goods

Can't Solve Inefficiency - because of duality, if someone manages to find a mechanism to solve the free rider problem in public goods (hence solving the inefficiency), that same mechanism could be used to enforce cartels for private goods (resulting in inefficiency)

Samuelson's View - argued if a good is not a pure private good (i.e., excludable and rival with no externalities), then it should be treated as a public good in that sum of MRS should equal MRT (i.e., FOC accounts for all costs/benefits)

Not So Extreme - need to distinguish solution to inefficiencies; flowers in your yard is not the same as national defense

Private Goods -

For Person *j* - $\text{MRS}^j_{x_1^j, x_2^j} = \text{MRT}_{x_1, x_2}$

For Person *k* - $\text{MRS}^k_{x_1^j, x_2^k} = 0$ ... i.e., how much person *k* is willing to give up (willingness to pay) of private good 2 in order for person *j* to consume more of private good 1 (nothing!)

Result - we can add these two equations together: $\text{MRS}^j_{x_1^j, x_2^j} + \text{MRS}^k_{x_1^j, x_2^k} = \text{MRT}_{x_1, x_2}$

Public Good - $\text{MRS}^j_{z_h^j, x_1^j} = \text{MRS}^j_{z_h^k, x_1^j}$ ... i.e., doesn't matter who consumes the public good (because everyone consumes the same amount)

Extremes - pure private good: $\text{MRS}^k_{x_1^j, x_2^k} = 0$; pure public good: $\text{MRS}^j_{z_h^j, x_1^j} = \text{MRS}^j_{z_h^k, x_1^j}$ ... other goods are between these extremes

Slutsky's Papers - focus on 1st and 2nd FTWE with Coasian bargaining

Paper 1 - "Production Externalities and Long-Run Equilibria: Bargaining and Pigovian Taxation"

Short Version - perfect information with free entry; 1st FTWE fails; Coasian bargaining doesn't work because 2nd order conditions fail

Scenario - 1 lake; 2 types of firms: fishing & chemical plant; chemicals dumped in lake affect fishing (but fishing doesn't impact chemical plant)

Simplifying Assumptions & Notation - don't really affect results, but simplify the math and make the result more obvious

1. Labor is the only input in both industries
2. Labor is numeraire (i.e., wage rate = 1)

3. $X =$ total production from chemical plants (polluting industry)
4. $Y =$ total production from fishing firms (polluted industry)
5. $F(x) =$ amount of labor needed for <u>individual</u> chemical firm to generate output x
6. Identical chemical firms $\therefore\ X = n_x x$
7. $G(y, X) =$ amount of labor needed for individual fishing firm to generate output y; assume for given level of output y, cost increases with X
   Atmospheric Externality - only total amount matters, not who or where it's produced
8. Second order conditions... $F(x)$ strictly convex with $\lim_{x \to 0} F(x) > 0$ (i.e., fixed cost of entry)... u-shaped average cost curve so we'll have limited number of chemical firms
9. SOC for fishing firms... $G(y, X)$ strictly convex with $\lim_{y \to 0} G(y, X) > 0$
10. Identical consumers with population n
11. **Quasilinear utility** - $u\left(\frac{X}{n}, \frac{Y}{n}\right) - \frac{L}{n}$... linear in labor so we can think of economy consisting of single individual: $\hat{u}(X,Y) - L$ (**standard trick** when focusing on production; eliminates income and substitution effects); assume $\hat{u}(X,Y)$ is strictly concave

Pareto Optimality - get unconstrained optimization by substituting supply of labor = demand for labor: $L = n_x F\left(\frac{X}{n_x}\right) + n_y G\left(\frac{Y}{n_y}, X\right)$

$$\max_{X,Y,n_x,n_y} W = \hat{u}(X,Y) - n_x F\left(\frac{X}{n_x}\right) - n_y G\left(\frac{Y}{n_y}, X\right)$$

Assumption - interior solution

Second Order Conditions - $\hat{u}(X,Y)$ is strictly concave; F and G are convex so the objective function is strictly concave.... **wrong!** no guarantee it's concave with respect to all four variables; check Hessian is negative definite:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 W}{\partial X^2} & \frac{\partial^2 W}{\partial X \partial Y} & \frac{\partial^2 W}{\partial X \partial n_x} & \frac{\partial^2 W}{\partial X \partial n_y} \\ \frac{\partial^2 W}{\partial Y \partial X} & \frac{\partial^2 W}{\partial Y^2} & \frac{\partial^2 W}{\partial Y \partial n_x} & \frac{\partial^2 W}{\partial Y \partial n_y} \\ \frac{\partial^2 W}{\partial n_x \partial X} & \frac{\partial^2 W}{\partial n_x \partial Y} & \frac{\partial^2 W}{\partial n_x^2} & \frac{\partial^2 W}{\partial n_x \partial n_y} \\ \frac{\partial^2 W}{\partial n_y \partial X} & \frac{\partial^2 W}{\partial n_y \partial Y} & \frac{\partial^2 W}{\partial n_y \partial n_x} & \frac{\partial^2 W}{\partial n_y^2} \end{bmatrix}$$

Need - each diagonal element < 0; determinant of 2x2 minors > 0; determinant of 3x3 minors > 0; determinant of matrix > 0

Assumption - some of these determinants are indeterminate (can be > 0 or < 0), but $\exists$ some $\hat{u}(X,Y)$, F and G such that W is concave

First Order Conditions -

$$\frac{\partial W}{\partial X} = \hat{u}_X(X,Y) - F'\left(\frac{X}{n_x}\right) - \overbrace{n_y G_X\left(\frac{Y}{n_y}, X\right)}^{\text{External Cost}} = 0 \quad (1)$$

$$\frac{\partial W}{\partial Y} = \hat{u}_Y(X,Y) - G_Y\left(\frac{Y}{n_y}, X\right) = 0 \quad (2)$$

$$\frac{\partial W}{\partial n_x} = \frac{X}{n_x}F'\left(\frac{X}{n_x}\right) - F\left(\frac{X}{n_x}\right) = xF'(x) - F(x) = 0 \quad (3)$$

$$\frac{\partial W}{\partial n_y} = \frac{Y}{n_y}G_Y\left(\frac{Y}{n_y}, X\right) - G\left(\frac{Y}{n_y}, X\right) = yG_Y(y,X) - G(y,X) = 0 \quad (4)$$

Note: for (3) and (4), we assumed n_x and n_y are continuous variables (so we could take derivatives), but they're actually integers

From (3): $xF'(x) = F(x) \Rightarrow F'(x) = \frac{F(x)}{x}$ ... marginal cost of chem = average cost

From (4): $yG_y(y,X) = G(y,X) \Rightarrow G_y(y,X) = \frac{G(y,X)}{y}$ ... MC of fish = AC

From (2): $\hat{u}_Y(X,Y) = G_y\left(\frac{Y}{n_y}, X\right)$... marginal benefit of fish = marginal cost

From (1): $\hat{u}_X(X,Y) = F'\left(\frac{X}{n_x}\right) + n_y G_X\left(\frac{Y}{n_y}, X\right)$... MB of chemicals = <u>social</u> MC (the sum of all the costs; similar to Samuelson condition [middle of p.3])

Competitive Equilibrium - "without getting into details"

Chemical Firm - $\max_x \; p_x x - F(x)$ ... FOC: (a) $p_x = F'(x)$

Fishing Firm - $\max_y \; p_y y - G(y,X)$ ... FOC: (b) $p_y = G_Y(y,X)$

Consumers - $\max_{X,Y,L} \; \hat{u}(X,Y) - L$ s.t. $p_x X + p_x Y = L$ ... embed constraint:

$\max_{X,Y} \; \hat{u}(X,Y) - p_x X - p_x Y$ ... FOCs: (c) $\hat{u}_X = p_x$; (d) $\hat{u}_Y = p_y$

Combine (b) and (d) to get (2)... $\hat{u}_Y = p_y = G_Y(y,X)$

Combine (a) and (c)... $\hat{u}_X = p_x = F'(x) \neq F'\left(\frac{X}{n_x}\right) + n_y G_X\left(\frac{Y}{n_y}, X\right)$ (1)... $\therefore$ the CE is not PO (1st FTWE fails)

Pigovian Solution - tax chemical firms: $\max_x \; p_x x - F(x) - t_x x$ ... FOC: (a) $p_x = F'(x) + t_x$; set $t_x = n_y G_X\left(\frac{Y}{n_y}, X\right)$ and CE will be PO

Coasian Solution - unclear who's involved because of free entry (Coase's paper only dealt with fixed number of participants); instead of bargaining, assume lake is owned by single individual who completely controls the lake (i.e. has property rights); assume no monitoring or enforcement costs for owner

Owners Revenue -

Optimal Fees - takes all profit from both firms: $I_x = p_x x - F(x)$ and

$I_y = p_y y - G(y, X) = p_y y - G(y, n_x x)$ (want to focus on output per firm)

$$\max_{n_x, n_y, x, y} R = n_x I_x + n_y I_y = n_x\left(p_x x - F(x)\right) + n_y\left(p_y y - G(y, n_x x)\right)$$

FOC - will compare these to PO FOCs on previous page

(i) $\dfrac{\partial R}{\partial n_x} = p_x x - F(x) - n_y x G_X(y, n_x x) = 0$

(ii) $\dfrac{\partial R}{\partial n_y} = p_y y - G(y, n_x x) = 0$

(iii) $\dfrac{\partial R}{\partial x} = \cancel{n_x}\left(p_x - F'(x)\right) - n_y \cancel{n_x} G_X(y, n_x x) = 0$

(iv) $\dfrac{\partial R}{\partial y} = \cancel{n_y}\left(p_y - G_y(y, n_x x)\right) = 0$

Combine (i) and (iii) to get (3)... from (i) $p_x - \dfrac{F(x)}{x} - n_y G_X(y, n_x x) = 0$; set that equal to (iii) $p_x - \dfrac{F(x)}{x} - n_y G_X(y, n_x x) = p_x - F'(x) - n_y G_X(y, n_x x) \Rightarrow$

$\dfrac{F(x)}{x} = F'(x)$ ... MC of chemicals = AC of chemicals

Combine (ii) and (iv) to get (4)... from (ii) $p_y - \dfrac{G(y, n_x x)}{y} = 0$; set that equal to

(iv) $p_y - \dfrac{G(y, n_x x)}{y} = p_y - G_y(y, n_x x) \Rightarrow \dfrac{G(y, X)}{y} = G_y(y, X)$

Combine (iii) with consumer FOC (c) to get (1) $\hat{u}_X = p_x = F'(x) + n_y G_X(y, X)$

Combine (iv) with consumer FOC (d) to get (2) $\hat{u}_y = p_y = G_y(y, X)$

∴ FOCs for Coasian bargaining (private owner of lake) are same as FOCs for PO solution, but...

SOC - full results in paper, but here' the basics

$\dfrac{\partial^2 R}{\partial n_y^2} = 0$ ... supposed to be < 0 (sufficient), but ≤ 0 is necessary... may be a problem

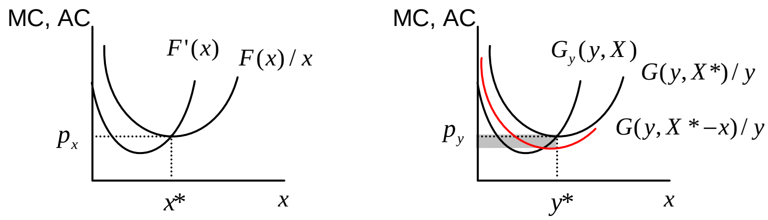
Check 2x2 minors... determinant must be ≥ 0:

$$\begin{vmatrix} \dfrac{\partial^2 R}{\partial n_x^2} & \dfrac{\partial^2 R}{\partial n_x \partial n_y} \\ \dfrac{\partial^2 R}{\partial n_x \partial n_y} & \dfrac{\partial^2 R}{\partial n_y^2} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 R}{\partial n_x^2} & \dfrac{\partial^2 R}{\partial n_x \partial n_y} \\ \dfrac{\partial^2 R}{\partial n_x \partial n_y} & 0 \end{vmatrix} = -\left(\dfrac{\partial^2 R}{\partial n_x \partial n_y}\right)^2 < 0$$

(as long as $\dfrac{\partial^2 R}{\partial n_x \partial n_y} = -x G_X \neq 0$)

∴ SOC fail and there's no assumption we can impose on F or G to make SOC hold

Problem - free entry results in zero profit for both firms so lake owner's revenue is zero; he can do better by removing a chemical firm which gives positive profits to fishing firms



Note: with fixed set of participants, both Coasian and Pigovian approaches work; problem with Coasian approach arises when there is not a fixed number of participants (i.e., doesn't work in long run)

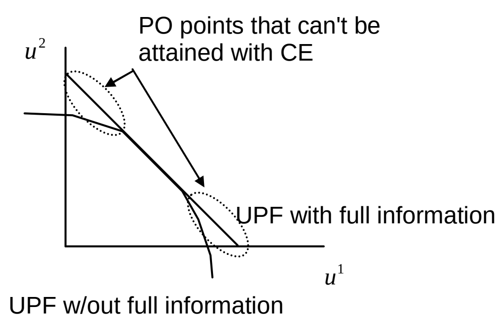
Paper 2 - "Private Information, Coasian Bargaining, and the Second Welfare Theorem"

Short Version - private information with no entry; 2nd FTWE fails; Coasian bargaining restricts government's ability to redistribute

Max social welfare subject to technology and self selection constraints

Private owner limits redistribution government can do ∴ 2nd FTWE fails (can't get to all PO points with CE)

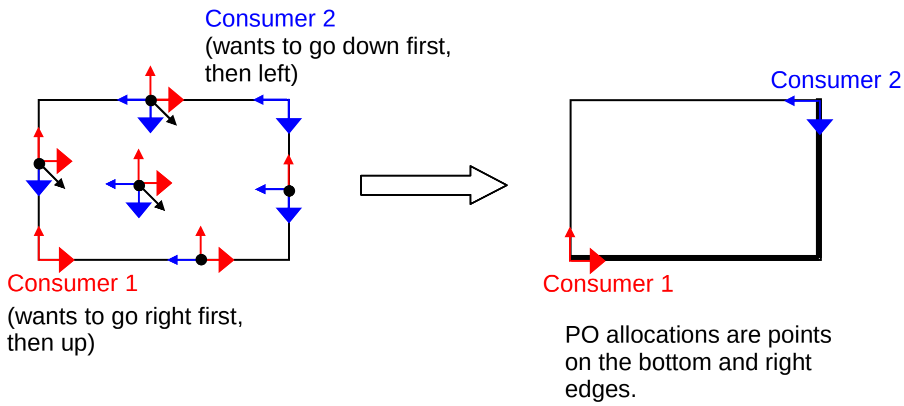
Even if private owner increases efficiency, will have people hurt who can lo longer get government redistribution



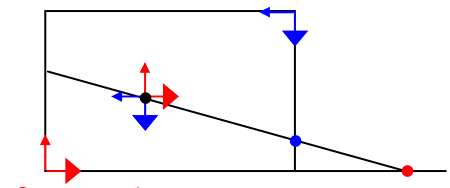
1a. Suppose two consumers have lexicographic preferences where person 1 ranks bundles by the level of commodity 1 and only considers the level of commodity two if bundles have the same amount of commodity 1. Person 2 does the reverse looking first at the level of commodity 2.
i) Find the set of Pareto efficient allocations in an Edgeworth box.
ii) Find the competitive equilibrium price ratio and the competitive equilibrium quantities for an arbitrary distribution of initial endowments.
iii) Does the Second Welfare Theorem hold in this economy?

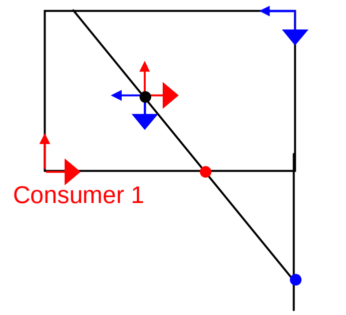
i) Put consumer 1 in the lower left, corner. We really can't draw indifference curves, but realize that consumer 1 is best off moving to the right first (i.e., more of good 1), then considers moving up (i.e., more of good 2) if he can't move right. Consumer 2 goes in the upper, right corner. Consumer 2 prefers good 2 so he is best off moving down first (i.e., more of good 2).

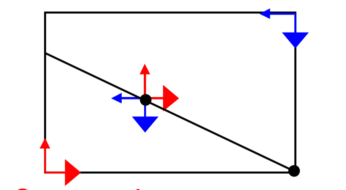
We don't have indifference curves to check tangency points. To find Pareto optimal points, we can strategically place a few points in the Edgeworth box and determine if those points are Pareto optimal (i.e., we can't make one person better off without making the other worse off). It's enough to check a point on each edge and one in the interior to determine where all the Pareto optimal points are. To keep from cluttering the picture (more than it already is), red arrows show the preferences for consumer 1 and blue arrows for consumer 2. The arrows are only included if it is feasible for the consumer increase consumption in that direction. A black arrow at a point shows a way to change the allocation so that both consumers are better off. Therefore, any point with a black arrow is not Pareto optimal. The end result is that all points on the bottom and right edges of the Edgeworth box are Pareto optimal. That is, all points where consumer 2 has all of good 2 or consumer 1 has all of good 1.



ii) A competitive equilibrium in this pure exchange example has two conditions: (a) both consumers maximize their utilities subject to their budget constraints (i.e., based on the current price ratio and endowments) and (b) market clearing (the total amount of each commodity that is consumed is equal to the total amount of the endowment for that commodity). Another way to look at a competitive equilibrium is to say that the point that each consumer wants to be at for a given price ratio is the same as the other consumer's point and is within the Edgeworth box (i.e., is feasible). There are three different types of initial endowment to consider: (1) an interior point, (2) on the bottom edge, (3) on the right edge. (Note: points on the left and top edge of the Edgeworth box will have the same result as an interior point.)







Interior Point - In order for consumer to 1 to maximize his utility, he wants to trade his entire endowment of good 2 to get as much of good 1 as the price ratio allows; ditto for consumer 2 with good 2. There are three different price ratios to consider (shown at right). First consider a price ratio that allows consumer 1 to trade all of his endowment of good 2 for more good 1 than is available to society. (Neither consumer knows what the limits are.) In this case, consumer 1 wants to be at the red point which is outside the Edgeworth box. As consumer 1 tries to get more of good 1, he'll end up driving up the price of good 1 relative to good 2 (i.e., the budget line will get steeper). Therefore, this price ratio will not result in a competitive equilibrium. Next, consider a price ratio that allows consumer 2 to trade all of his endowment of good 1 for more good 2 than is available to society. In this case, consumer 2 will want to be at the blue point which is outside the Edgeworth box. As in the previous problem, consumer 2's attempt to attain consumption that is not feasible will drive up the price for good 2 (making the budget line more shallow). Therefore, this will not be a competitive equilibrium. Finally, consider a price ratio that connects the endowment point to the bottom, right corner of the Edgeworth box. In this case, consumer 1 trades his entire endowment of good 2 to gain the remaining amount of good 1 in society. Similarly, consumer 2 trades his entire endowment of good 1 for the remaining amount of good 2 in society. The point that both consumers want to be at is the same and is feasible. Therefore, the bottom, right corner of the Edgeworth box is the only competitive equilibrium for an endowment in the interior (or left and top edge).

Summary - CE is the bottom, right corner of the Edgeworth box and the price ratio is line between the endowment point and the CE point.

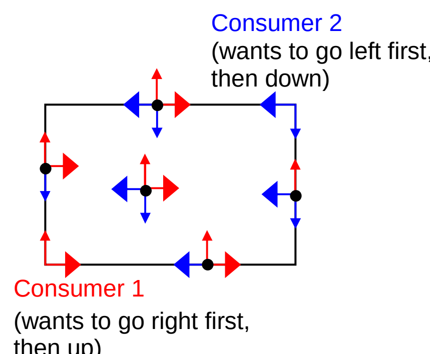
Bottom Edge - At any endowment point on the bottom edge, consumer 1 wants to consume more of good 1, but he has no good 2 to trade for it so no price ratio will increase his utility. Consumer 2, however, will have some of good 1 to trade away for more good 2 based on the price ratio. Realize this would be the same as starting at the red point in the middle graph above. Any price ratio except one with the price of good 1 equal to zero would have consumer

2 wanting to get more good 2 (which is not feasible). At a price of zero for good 1, however, we end up with Arrow's exceptional case. Both consumers have unbounded utility maximization problems so neither can actually maximize utility. Therefore, at the bottom edge, there is no equilibrium (except the right corner discussed in the previous section).

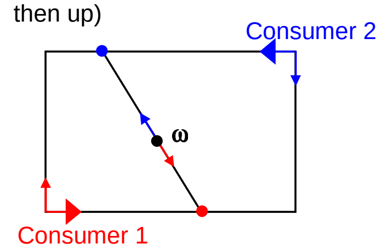
Right Edge - The right edge follows the same argument as the bottom edge, except we're dealing with consumer 2 having no endowment of good 1 to trade and we'll end up with the price of good 2 equal to zero. Again, we face Arrow's exceptional case with each consumer having unbounded utility maximization problems. Therefore, there is no equilibrium on the right edge (except the bottom corner discussed in the first section).

iii. The second welfare theorem states that given any Pareto optimal allocation, there exists a price ratio and an initial endowment such that a competitive equilibrium results in the same allocation. As part ii just covered, the only competitive equilibrium in this scenario is the bottom, right corner of the Edgeworth box. Since there are other points that are Pareto optimal, it would appear the second welfare theorem does not hold. Recall, however, that the assumptions we made to prove the second welfare theorem involved preferences being complete, convex, transitive, locally nonsatiated, and continuous. Lexicographic preferences in this problem violate continuity.

1b. Now assume lexicographic preferences where both consumers care first about the level of commodity 1 and only for bundles with the same amount of commodity one do they look at the level of commodity 2. Find the set of Pareto optimal allocations in an Edgeworth box. Are there any distributions of initial endowments at which a competitive equilibrium exists?



Apply the same strategy as part 1a (i.e., put points on each edge and the interior). In this case, both consumers' preferences are exactly opposite so one can never be made better without hurting the other. That means every point is Pareto optimal.



Given endowment ω and any price ratio, consumer 1 will want to trade to get as much good 1 as possible. So will consumer 2. Note in the lower picture that consumer 1 maximizes utility at the red point and consumer 2 does so at the blue point. These points are not the same (and won't be regardless of the price ratio) so a competitive equilibrium does not exist.

2. Consider a two-person, two-commodity pure exchange competitive economy,. The consumers' utility functions are:

$$u^1(\mathbf{x}^1) = x_1^1 x_2^1 + 12x_1^1 + 3x_2^1 \quad \text{and} \quad u^2(\mathbf{x}^2) = x_1^2 x_2^2 + 8x_1^2 + 9x_2^2$$

(Superscripts refer to consumers and subscripts to goods.) Consumer 1 has an endowment $\boldsymbol{\omega}^1 = (8,30)$ and consumer 2 has an endowment $\boldsymbol{\omega}^2 = (10,10)$
Determine the excess demand functions for the two consumers. Find the equilibrium price ratio for this economy.

Consumer 1: $\max_{x_1^1, x_2^1} u^1(\mathbf{x}^1) = x_1^1 x_2^1 + 12x_1^1 + 3x_2^1$ s.t. $p_1 x_1^1 + p_2 x_2^1 \le 8p_1 + 30p_2$

$$L^1 = x_1^1 x_2^1 + 12x_1^1 + 3x_2^1 - \lambda^1 (p_1 x_1^1 + p_2 x_2^1 - 8p_1 - 30p_2)$$

(1.1) $\dfrac{\partial L^1}{\partial x_1^1} = x_2^1 + 12 - \lambda^1 p_1 = 0$

(1.2) $\dfrac{\partial L^1}{\partial x_2^1} = x_1^1 + 3 - \lambda^1 p_2 = 0$

(1.3) $-\dfrac{\partial L^1}{\partial \lambda^1} = p_1 x_1^1 + p_2 x_2^1 - 8p_1 - 30p_2 = 0$

Solve (1.1) and (1.2) for λ^1 and set them equal to each other:

$$\lambda^1 = \frac{x_2^1 + 12}{p_1} = \frac{x_1^1 + 3}{p_2}$$

Now solve that for x_2^1 in terms of x_1^1:

$$x_2^1 = \frac{p_1 x_1^1 + 3p_1}{p_2} - 12$$

Substitute this into (1.3):

$$p_1 x_1^1 + p_2 \left(\frac{p_1 x_1^1 + 3p_1}{p_2} - 12 \right) - 8p_1 - 30p_2 = 2p_1 x_1^1 - 5p_1 - 42p_2 = 0$$

Now solve that for x_1^1:

$$\boxed{x_1^1 = \frac{5}{2} + 21\frac{p_2}{p_1}}$$

Plug that back into the x_2^1 equation:

$$x_2^1 = \frac{p_1\left(\frac{5}{2} + 21\frac{p_2}{p_1}\right) + 3p_1}{p_2} - 12 = \boxed{x_2^1 = \frac{11p_1}{2p_2} + 9}$$

Consumer 2: $\max_{x_1^2, x_2^2} u^2(\mathbf{x}^2) = x_1^2 x_2^2 + 8x_1^2 + 9x_2^2$ s.t. $p_1 x_1^2 + p_2 x_2^2 \le 10p_1 + 10p_2$

$$L^2 = x_1^2 x_2^2 + 8x_1^2 + 9x_2^2 - \lambda^2 (p_1 x_1^2 + p_2 x_2^2 - 10p_1 - 10p_2)$$

(2.1) $\dfrac{\partial L^2}{\partial x_1^2} = x_2^2 + 8 - \lambda^2 p_1 = 0$

(2.2) $\dfrac{\partial L^2}{\partial x_2^2} = x_1^2 + 9 - \lambda^2 p_2 = 0$

(2.3) $-\dfrac{\partial L^2}{\partial \lambda^2} = p_1 x_1^2 + p_2 x_2^2 - 10 p_1 - 10 p_2 = 0$

Solve (2.1) and (2.2) for λ^2 and set them equal to each other:

$$\lambda^2 = \frac{x_2^2 + 8}{p_1} = \frac{x_1^2 + 9}{p_2}$$

Now solve that for x_2^2 in terms of x_1^2:

$$x_2^2 = \frac{p_1 x_1^2 + 9 p_1}{p_2} - 8$$

Substitute this into (2.3):

$$p_1 x_1^2 + p_2\left(\frac{p_1 x_1^2 + 9 p_1}{p_2} - 8\right) - 10 p_1 - 10 p_2 = 2 p_1 x_1^2 - p_1 - 18 p_2 = 0$$

Now solve that for x_1^2:

$$\boxed{x_1^2 = \frac{1}{2} + 9\frac{p_2}{p_1}}$$

Plug that back into the x_2^2 equation:

$$x_2^2 = \frac{p_1\left(\frac{1}{2} + 9\frac{p_2}{p_1}\right) + 9 p_1}{p_2} - 8 = \quad \boxed{x_2^2 = \frac{19 p_1}{2 p_2} + 1}$$

Excess demands are:

$$x_1^1 + x_1^2 - \omega_1^1 - \omega_1^2 = \left(\frac{5}{2} + 21\frac{p_2}{p_1}\right) + \left(\frac{1}{2} + 9\frac{p_2}{p_1}\right) - 8 - 10 \;\Rightarrow\; \text{Good 1: } \boxed{z^1 = 30\frac{p_2}{p_1} - 15}$$

$$x_2^1 + x_2^2 - \omega_2^1 - \omega_2^2 = \left(\frac{11 p_1}{2 p_2} + 9\right) + \left(\frac{19 p_1}{2 p_2} + 1\right) - 30 - 10 \;\Rightarrow\; \text{Good 2: } \boxed{z^2 = 15\frac{p_1}{p_2} - 30}$$

Solve for the price ratio by setting excess demands to zero (Walras' Law)... only need one of these, but did both to check my work:

$$30\frac{p_2}{p_1} - 15 = 0 \;\Rightarrow\; \frac{p_2}{p_1} = \frac{15}{30} = \frac{1}{2}$$

$$\frac{30 p_1}{2 p_2} - 30 = 0 \;\Rightarrow\; \frac{p_1}{p_2} = \frac{30}{15} = 2 \qquad \boxed{\frac{p_1}{p_2} = 2}$$

3. Consider an exchange economy with two identical consumers. Each has the utility function $u^1 = x^a y^{1-a}$ for $0 < a < 1$. Society has 10 units of good x and 10 units of good y. Find a set of endowments $\boldsymbol{\omega}^1$ and $\boldsymbol{\omega}^2$ with $\boldsymbol{\omega}^1 \neq \boldsymbol{\omega}^2$ and a Walrasian equilibrium price ratio which will support equilibrium consumption bundles of (5,5) for each consumer.

Each consumer must maximize his utility:

$$\max_{x,y} u^i = x^a y^{1-a} \text{ s.t. } p_x x + p_y y \leq p_x \omega_x^i + p_y \omega_y^i$$

We know the price ratio will be equal to the marginal rate of substitution:

$$\frac{p_x}{p_y} = \frac{\partial u^i / \partial x}{\partial u^i / \partial x} = \frac{a x^{a-1} y^{1-a}}{(1-a) x^a y^{-a}} = \frac{a}{1-a} \frac{y}{x}$$

At the point (5,5), the price ratio becomes: $\frac{p_x}{p_y} = \frac{a}{1-a}\frac{5}{5} =$ $\boxed{\frac{p_x}{p_y} = \frac{a}{1-a}}$

At (5,5) both indifference curves and the budget line are tangent. In order to attain (5,5) as a competitive equilibrium, all we have to do is set the endowment point on the budget line. Therefore, the set of endowments are all those points on the line through point (5,5) with slope -*a*/(1 - *a*) ... limited to the total amount of each good being 10 (i.e., $\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2 = (10{,}10)$).

Equation of a line: $\omega_2^1 = \frac{-a}{1-a}\omega_1^1 + b$

Given point (5,5) solve for b: $5 = \frac{-a}{1-a}5 + b \;\Rightarrow\; b = 5 + \frac{5a}{1-a} = \frac{5-5a}{1-a} + \frac{5a}{1-a} = \frac{5}{1-a}$

So the set of endowments is given by:

$$\boxed{\begin{aligned} \omega_2^1 &= \frac{-a}{1-a}\omega_1^1 + \frac{5}{1-a} \\ \omega_1^2 &= 10 - \omega_1^1 \\ \omega_2^2 &= 10 - \omega_2^1 \end{aligned}}$$

Documentation.

I went over all three problems with Prof Slutsky. He told me my method and answer for 1ai was correct. He told me the bottom, right corner was the competitive equilibrium in 1aii (and explained why). He explained why competitive equilibrium doesn't exist in 1b. He told me I only needed one equation and one unknown (price ratio) in 2. He pointed out that 3 asked for a set of endowments, not a specific point.

Christine caught 2 errors in my work. One in problem 2 where I didn't divide the 18 by 2 when I solved for x_1^2. The other was on problem 3 where I didn't cancel correctly to get p_x/p_y. Guille caught more errors: confused left and right in 1a and forgot the negative sign on the slope in problem 3.

4. A standard modification of the basic general equilibrium model is to allow dependent consumer preferences (externalities). Assume two individuals whose utilities each depend upon the consumption bundle chosen by the other. Let $P \subseteq R^n$ be the unit price simplex and $X^i \subseteq R^n$, $i = 1, 2$, be the consumption sets for individuals 1 and 2 and let $\boldsymbol{\omega}^i \in R^n$, $i = 1, 2$, be the initial endowment vectors. Define a mapping from $P \times X^1 \times X^2$ into itself whose fixed point is an equilibrium in this economy. What properties must this mapping satisfy for the fixed point to exist? In general, how would you show that these properties are satisfied?

Let $D^i(\mathbf{p}, \mathbf{x}^j)$ be the utility maximizing demand for consumer i facing prices $\mathbf{p}$ and consumption $\mathbf{x}^j$ by consumer j

Mapping - given any $(\mathbf{p}, \mathbf{x}^1, \mathbf{x}^2) \in P \times X^1 \times X^2$, define $(\hat{\mathbf{p}}, \hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2)$ by

$$\hat{\mathbf{x}}^1 = D^1(\mathbf{p}, \mathbf{x}^2)$$
$$\hat{\mathbf{x}}^2 = D^2(\mathbf{p}, \mathbf{x}^1)$$
$$\hat{p}_j = \frac{p_j + \max\left[0, \sum_{k=1}^{m}(\mathbf{x}^k - \boldsymbol{\omega}^k)\right]}{1 + \sum_{l=1}^{n}\max\left[0, \sum_{k=1}^{m}(x_l^k - \boldsymbol{\omega}_l^k)\right]}$$

Kakutani Fixed Point Theorem - if $f : S \to 2^S$ is compact valued, convex valued, and upper hemi continuous, and S is compact and convex, then $\exists\ \mathbf{x}^* \in S$ with $\mathbf{x}^* \in F(\mathbf{x}^*)$

Based on the Kakutani FPT, we need to show:

1a. $D^i(\mathbf{p}, \mathbf{x}^j)$ compact valued - follows from utility optimization constraint on $\mathbf{x}^i$: $0 \le \mathbf{x}^i \le \hat{k}$

1b. $\hat{p}_j$ compact valued - follows from the equation in the mapping ($\hat{p}_j \in [0,1]$)

2a. $D^i(\mathbf{p}, \mathbf{x}^j)$ convex valued - follows from assumption that preferences are convex

2b. $\hat{p}_j$ convex valued - follows from the equation in the mapping; sum of excess demands is convex valued so the maximizations in the equation are convex; p_j and 1 are single values ($\therefore$ convex); adding convex values and dividing by convex values ($\neq 0$) gives a convex value

3a. $D^i(\mathbf{p}, \mathbf{x}^j)$ upper hemi continuous - from Berge Maximum Theorem

3b. $\hat{p}_j$ upper hemi continuous - follows form the equation in the mapping; sum of excess demands is UHC (from 3a); maximizations in the equation are continuous; p_j and 1 are single values ($\therefore$ continuous); adding continuous values and dividing by continuous values ($\neq 0$) gives a continuous value

4. $P \times X^1 \times X^2$ compact - Cartesian product of compact sets is a compact set:
5. $P \times X^1 \times X^2$ convex - Cartesian product of convex sets is a convex set:

   P compact & convex - $P = \left\{ p : p_j \geq 0 \,\&\, \sum_{i=1}^{n} p_j = 1 \right\}$

   X^i compact & convex - follows from demands being compact valued and convex valued (as long as we have a $p_j \neq 0$)

5. Consider an economy in which excess demands for commodities as functions of prices are denoted by $E_i = E_i(\mathbf{p})$, $i = 1, 2, ..., n$. Let $E = \max(|E_i|)$, that is E is the maximum of the absolute values of the E_i. Consider a mapping from $P \times E$ (Cartesian product of price simplex and excess demand space) into $\hat{P} \times \hat{E}$ defined by:

$$\hat{E}_i = E_i(\mathbf{p}), \; i = 1, 2, ..., n$$

$$\hat{p}_i = \frac{p_i + E}{1 + nE}, \; i = 1, 2, ..., n$$

Under what conditions will this mapping have a fixed point? Show that if an equilibrium existed for this economy, it would be a fixed point of this mapping. Explain why showing that this mapping has a fixed point does not prove that there is an equilibrium for the economy.

Kakutani Fixed Point Theorem - if $f : S \rightarrow 2^S$ is compact valued, convex valued, and upper hemi continuous, and S is compact and convex, then $\exists\ \mathbf{x}^* \in S$ with $\mathbf{x}^* \in F(\mathbf{x}^*)$

The mapping must be compact valued, convex valued, and upper hemi continuous, and $\hat{P} \times \hat{E}$ must be compact and convex for the mapping to have a fixed point.

Assume $(\mathbf{p}^*, \mathbf{x}^*) \in P \times E$ is an equilibrium

Given that this is an equilibrium point, the excess demand for prices $\mathbf{p}^*$ is zero for each good. That means $E_i(\mathbf{p}^*) = 0$ and $E = \max(|E_i|) = 0$.

Putting this point into the mapping yields the same point:

$$\hat{E}_i = E_i(\mathbf{p}^*) = 0, \; i = 1, 2, ..., n$$

$$\hat{p}_i = \frac{p_i^* + 0}{1 + n(0)} = p_i^*, \; i = 1, 2, ..., n$$

$\therefore$ $(\mathbf{p}^*, \mathbf{x}^*)$ maps to itself (i.e., it is a fixed point of this mapping)

Needed to show: mapping ⇒ fixed point ⇒ equilibrium
Showed: equilibrium ⇒ fixed point
That's not the same thing!
It is possible to find another point that is not an equilibrium:

Pick any quantity vector so that we have a fixed maximum excess demand $\tilde{E}$

Define $\tilde{\mathbf{p}}$ as the price vector that maps to itself given $\tilde{E}$: $\tilde{p}_i = \dfrac{\tilde{p}_i + \tilde{E}}{1 + n\tilde{E}}$

Solve this for $\tilde{p}_i$:

$$\tilde{p}_i(1 + n\tilde{E}) = \tilde{p}_i + \tilde{E}$$
$$\cancel{\tilde{p}_i} + \tilde{p}_i n\tilde{E} = \cancel{\tilde{p}_i} + \tilde{E}$$
$$\tilde{p}_i = \frac{\tilde{E}}{n\tilde{E}} = \frac{1}{n}$$

∴ if we set $\tilde{\mathbf{p}}$ to have all prices equal to 1/n (which will be in the price simplex since each $0 < \tilde{p}_j < 1$ and they sum to 1) and choose any allocation that results in the same excess demands as would be generated by prices $\tilde{\mathbf{p}}$, that point maps to itself. In general this point would not be an equilibrium.

Another candidate point:

Pick the point $(\mathbf{p}^*, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}}^i = \boldsymbol{\omega}^i$

As before, we have excess demand for prices $\mathbf{p}^*$ equal to zero for each good and we get $E_i(\mathbf{p}^*) = 0$ and $E = \max(|E_i|) = 0$. Realize that if each consumer consumes his endowment, we have zero excess demand: Since each consumer is consuming his endowment, excess demand will be zero for each good. That means $(\mathbf{p}^*, \tilde{\mathbf{x}})$ maps to itself. In general the endowment point is not an equilibrium.

Documentation.

Prof Slutsky gave the mapping in problem 4 in class. He also told me how to explain $\hat{p}_j$ is UHC. For problem 5, Prof Slutsky liked my idea of using the optimal prices with another quantity vector, but he wasn't sure it would work. He told me to fix E and solve for $\hat{p}_i$.