Solving Differential Equations

"Solving" means we can get ride of the derivative terms

1) Problem: $\boxed{y' = \frac{dy}{dt} = \frac{f(t)}{g(y)}}$

   We can rearrange terms to get $g(y)dy = f(t)dt$

   Key step is that we can now integrate both sides: $\int g(y)dy = \int f(t)dt$

2) Problem: $\boxed{y' = \frac{dy}{dt} = \frac{-M(t,y)}{N(t,y)}}$

   Similar to case 1, but now both terms are functions of both variables
   We can rearrange terms to get $N(t,y)dy = -M(t,y)dt$ or $M(t,y)dt + N(t,y)dy = 0$

   Now we'd like to find a function $f(t,y)$ such that $\frac{\partial f}{\partial t} = M(t,y)$ and $\frac{\partial f}{\partial y} = N(t,y)$

   If the function exists, then $f(t,y) = c$ (any constant) solves y'

   $M(t,y)dt + N(t,y)dy = 0$ is called an <u>exact differential</u>

   (because if we totally differentiate $f(t,y) = c$, we get $f_t dt + f_y dy = 0$ which is the same as $M(t,y)dt + N(t,y)dy = 0$)

   To test for the existence of an exact differential check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, which is the same as

   $\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial^2 f}{\partial y \partial t}$ (Young's Theorem)

3) Problem: $\boxed{y'(t) + Py(t) = Q}$

   Multiply everything by e^{Pt}: $e^{Pt}y'(t) + Pe^{Pt}y(t) = e^{Pt}Q$

   Note that $\frac{d}{dt}\left(e^{Pt}y(t)\right) = e^{Pt}y'(t) + Pe^{Pt}y(t)$

   Substitute that in the left side: $\frac{d}{dt}\left(e^{Pt}y(t)\right) = e^{Pt}Q$

   Integrate both sides: $\int \frac{d}{dt}\left(e^{Pt}y(t)\right)dt = \int e^{Pt}Qdt \Rightarrow e^{Pt}y(t) = \frac{e^{Pt}Q}{P} + c$ (c is any constant)

   Now solve for $y(t)$: $y(t) = \frac{Q}{P} + ce^{-Pt}$

3a) Problem: $y'(t) + Py(t) = Q(t)$

Same as 3, except we can't simplify the right hand side like we did; instead after we integrate both sides we end up with:

$$\int \frac{d}{dt}\left(e^{Pt} y(t)\right) dt = \int e^{Pt} Q dt \quad \Rightarrow \quad e^{Pt} y(t) = \int e^{Pt} Q dt + c$$ (c is any constant)

Now solve for $y(t)$: $y(t) = e^{-Pt}\left[\int e^{Pt} Q dt + c\right]$

4) Problem: $\boxed{\begin{aligned} \frac{dx}{dt} &= a_1 x + b_1 y \\ \frac{dy}{dt} &= a_2 x + b_2 y \end{aligned}}$

The solution to the homogeneous system is:

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$
$$y(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$$

where r_1 and r_2 are the eigenvalues of the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

$B_1 = \frac{(r_1 - a_1)A_1}{b_1}$ and $B_2 = \frac{(r_2 - a_1)A_2}{b_1}$

There must be two initial/terminal conditions in order to determine the constants A_1 and A_2

5) Problem: $\boxed{x''(t) + P(t)x'(t) + Q(t)x(t) = R(t)}$ (complete equation)

Reduced Equation: $x''(t) + P(t)x'(t) + Q(t)x(t) = 0$

Theorem 1 - general solution to the complete equation is the sum of any particular solution of the complete equation and the general solution of the reduced equation

Theorem 2 - any solution of the reduced equation on $t_0 \le t \le t_1$ can be expressed as a linear combination of any two solutions x_1 and x_2 that are linearly independent:

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

Special Case - $P(t) = A$ and $Q(t) = B$ are constant

Characteristic Equation: $ce^{rt}(r^2 + Ar + B) = 0$ (c and r are constants)

$\therefore\ r^2 + Ar + B = 0$ ... using quadratic formula: $r_1, r_2 = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$

Case 1 - $A^2 > 4B$

r_1 and r_2 are real and distinct and general solution to reduced equation is

$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ (c_1 and c_2 are arbitrary constants)

Case 2 - $A^2 = 4B$

$r_1 = r_2 = -A/2$; there is only one solution: $x(t) = e^{rt}(c_1 + c_2 t)$

Case 3 - $A^2 < 4B$

r_1 and r_2 are complex conjugates... too complicated (see p.334 of KS)

Calculus of Variations

Techniques to Cover

1. Calculus of Variations - analytical method for solving problems over continuous time or distribution; solution is function (not single value or range of values)
2. Optimal Control Theory
3. Dynamic Programming

Sample Problems

Exhaustible Resources (e.g., drilling for oil)
Growth
Asymmetric Information

What is a Dynamic Optimization Problem?

Static Problem - x is sales level; $f(x)$ is profit on sales

$$\text{Max } f(x) \text{ s.t. } x \geq 0$$

Solution - 1st order condition: $f'(x^*) = 0$; 2nd order condition: $f''(x^*) \leq 0$; solve first order condition for x^*

Multi-Period Problem - x_t is sale in period t; $f(t,x_t)$ is profit on sales in period t

$$\text{Max } \sum_{t=1}^{T} f(t, x_t) \text{ s.t. } x_t \geq 0,\ t = 1, ..., T$$

Solution - not dynamic because decisions are independent (x_t doesn't affect profit from other periods); can solve same 1st order condition of static problem individually for each x_t^*

Getting Dynamic - suppose changing x from period to period has a cost

$$\text{Max } \sum_{t=1}^{T} f(t, x_t, x_{t-1}) \text{ s.t. } x_t \geq 0,\ t = 1, ..., T$$

More Data - need to specify x_0
Not Continuous - this isn't continuous, but you're more likely to encounter this because (a) many studies use discrete multiperiod model to examine dynamic issues; (b) there's not much truly continuous data; and (c) it's easier to work with than continuous case
Solution - 1st order conditions don't separate and must be solved simultaneously
Backward Induction - similar to game theory; take x_{t-1} as given and solve
Max $f(x_{t-1},x_t)$... 1st order condition: $f'(x_{t-1},x_t^*) = 0$
Last 2 periods: Max $f_{t-1}(x_{t-2},x_{t-1}) + f_t(x_{t-1})$... already solved $f_t(x_{t-1},x_t^*)$ to get x_t^* as function of x_{t-1} so we can view f_t as function of x_{t-1} only
1st order condition: $f'_{x_{t-1}}(x_{t-2},x_{t-1}^*) + f'_{x_{t-1}}(x_{t-1}^*) = 0$; solves x_{t-1}^* in terms of x_{t-2}
Repeat until you get to period 1

Getting Continuous - do multi-period problem with continuous time

$$\text{Max} \int_0^T f(t, x_t)dt \text{ s.t. } x(t) \geq 0$$

Not Dynamic - profit at time t is completely determined by $x(t)$
Solution - cornerwise optimization; 1st order condition: $f_x'(t, x^*(t)) = 0$... solve for $x^*(t)$

Real Dynamic Optimization Problem - continuous time for problem with cost for changing x from period to period

$$\text{Max} \int_0^T f(t, x_t, x_{t-1})dt \text{ s.t. } x(t) \geq 0 \text{ and } x(0) = x_0$$

Solution - we'll get to this later

Production Example - firm receives order for B units of product to be delivered by time T;
$x(t)$ = inventory at time t
$x'(t)$ = change in inventory (i.e., production rate)
Unit production cost rises linearly with production rate: $[c_1x'(t)]x'(t) = c_1[x'(t)]^2$
Unit cost of holding Inventory is constant: $c_2x(t)$
Objective - minimize cost:

$$\text{Min} \int_0^T \left[c_1[x'(t)]^2 + c_2x(t)\right]dt \text{ s.t. } x'(t) \geq 0,\ x(0) = 0,\ x(T) = B$$

Consumption Example - production function based on capital stock, K; items produced are either consumed or used to replenish (or increase) capital stock at rate $K' = dK/dt$
Simple Version - have utility function based on consumption, $C(t)$; get to $C(t)$ by relating it to production function: $F(K(t)) = C(t) + K'(t)$... total output available at time t = consumption at time t plus investment at time t

$$\text{Max} \int_0^T U(C(t))dt = \int_0^T U(F(K(t)) - K'(t))dt \text{ s.t. } K'(t) \geq 0 \text{ and } K(0) = K_0$$

Add Depreciation - assume capital decays at constant rate b; now total available product is $F(K(t)) = C(t) + K'(t) + bK(t)$... in discrete case it would depend on whether we depreciate at beginning or end of period, but in continuous case we don't have to worry about it
Add Discounting - look at general case first; start with A_0 dollars invested at interest rate r per year;
$A(1) = A_0(1 + r)$; $A(2) = A_0(1 + r)^2$; $A(3) = A_0(1 + r)^3$
Compounding - use same annual rate, but compound m times per year:
$A(1) = A_0(1 + r/m)^m$; $A(t) = A_0(1 + r/m)^{mt}$
Continuous Compounding - $\lim_{m\to\infty}(1 + r/m)^{mt} = e^{rt}$
Present Value - same as discounted value; B dollars in t years... $B = xe^{rt}$ $\therefore$ B has discounted value of $x = Be^{-rt}$

$$\text{Max} \int_0^T e^{-rt}U(F(K(t)) - K'(t) - bK(t))dt \text{ s.t. } K'(t) \geq 0 \text{ and } K(0) = K_0$$

How to Solve (Section I.3)

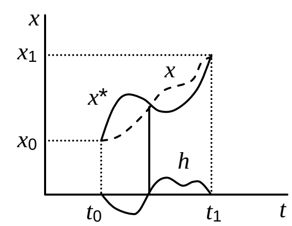
Simplest continuous, dynamic optimization problem:

$$\text{Max} \int_{t_0}^{t_1} F(t, x(t), x'(t))dt \quad \text{s.t. } x(t_0) = x_0 \text{ and } x(t_1) = x_1$$

Assumptions - F is continuous in its three arguments (t, x, x') and has continuous partial derivatives with respect to the second and third (x & x')

Independent - F is viewed as function of three independent arguments

Admissible Function - "feasible" solution to optimization problem; a function $x(t)$ that is continuously differentiable defined on the interval $[t_0,t_1]$ satisfying the fixed endpoint conditions ($x(t_0) = x_0$ and $x(t_1) = x_1$)



Optimal Solution - assume $x^*(t)$ solves the problem

Difference Function - assume $x(t)$ is some other admissible (feasible) function; the difference function is $h(t) = x(t) - x^*(t)$; **Note:** $h(t_0) = h(t_1) = 0$ because both $x(t)$ and $x^*(t)$ are admissible so they have the same value at t_0 and t_1

General Admissible Function - $g(t) = x^*(t) + ah(t)$ must be admissible for all a because it is continuously differentiable and satisfies the fixed endpoint conditions: $g(t_0) = x^*(t_0) + ah(t_0) = x_0 + a(0) = x_0$ and $g(t_1) = x^*(t_1) + ah(t_1) = x_1 + a(0) = x_1$

General is Optimal - since $x^*(t)$ is optimal, $g(t)$ will be optimal if $a = 0$

General as Function of *a* - since $x^*(t)$ and $h(t)$ are fixed, we can look at g as a function of a... $g(a)$; we just said g is optimal if $a = 0$; we can combine that with the first order condition to get $g'(0) = 0$

Note: $h(t)$ is fixed because we picked a specific $x(t)$ to generate it; the $x(t)$ we picked is fixed after we pick it, but we could pick any feasible $x(t)$ ∴ $h(t)$ can be any function as long as it comes from a feasible $x(t)$; the only restriction on $h(t)$ is $h(t_0) = h(t_1) = 0$ which was mentioned earlier

Put it Back - now plug $g(t)$ into original objective: $g(a) \equiv \int_{t_0}^{t_1} F(t, g(t), g'(t))dt$

Substitute $g(t) = x^*(t) + ah(t)$ and $g'(t) = x^{*\prime}(t) + ah'(t)$:

$$g(a) \equiv \int_{t_0}^{t_1} F(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))dt$$

Trick 1 - $g(a)$ is nice, but we want to work with the first order condition $g'(0) = 0$; to work with $g'(a)$ instead of $g(a)$ we need to differentiate the identity above with respect to a:

Note: in order to save space, $F(t,x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))$ will just be written as F

$$g'(a) \equiv \int_{t_0}^{t_1} \frac{dF}{da} dt$$

Trick 2 - we are totally differentiating F by a:

$$\frac{dF}{da} = F_t \frac{dt}{da} + F_x \frac{d(x^*(t) + ah(t))}{da} + F_{x'} \frac{d(x^{*\prime}(t) + ah'(t))}{da} = F_x h(t) + F_{x'} h'(t)$$

Note: F_i is the partial derivative of F with wrt the i^{th} argument; here we have i = 1, 2, 3 or we can use $i = t, x, x'$

Note: $dt/da = 0$

Now we can rewrite g'(a):

$$g'(a) \equiv \int_{t_0}^{t_1} \left[F_x h(t) + F_{x'} h'(t)\right] dt$$

Trick 3 - we want to get the two terms to have the same $h(t)$ term so we need to get ride of the $h'(t)$ in the second term; we'll do that by breaking up the integral, solving the second part and then recombining... in terms Len can understand here's what we're going to do:

$$\int (a+b)dt = \int adt + \int bdt$$

Then solve $\int bdt = \int cdt$

Now we have $\int adt + \int bdt = \int adt + \int cdt = \int (a+c)dt$

Note: this requires the range on the integrals to be the same

Applying the first part of the trick:

$$g'(a) \equiv \int_{t_0}^{t_1} F_x h(t)dt + \int_{t_0}^{t_1} F_{x'} h'(t)dt$$

Trick 4 - this is supposed to be "simple algebra" according to Dai... I'm glad I didn't go to his school! trick 4 is **integration by parts** in order to accomplish trick 3

$$\int udv = uv - \int vdu$$

To apply that to the second term in $g'(a)$, we'll use:

$u = F_x'$

$dv = h'(t)dt = \dfrac{dh(t)}{dt}dt = dh(t) \Rightarrow v = h(t)$

Now using integration by parts:

$$\int_{t_0}^{t_1} F_{x'} h'(t)dt = F_{x'} h(t)\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} h(t)dF_{x'} = \left[F_{x'} h(t_1) - F_{x'} h(t_0)\right] - \int_{t_0}^{t_1} h(t)dF_{x'} = -\int_{t_0}^{t_1} h(t)dF_{x'}$$

Note: the $Fx'h(t)$ terms go away because $h(t_0) = h(t_1) = 0$

Trick 4.5 - add dt/dt to the integral so we can combine it with the first term in $g'(a)$

$$\int_{t_0}^{t_1} F_{x'} h'(t)dt = -\int_{t_0}^{t_1} h(t)\frac{dF_{x'}}{dt}dt$$

Now add that back into $g'(a)$

$$g'(a) \equiv \int_{t_0}^{t_1} F_x h(t)dt + \int_{t_0}^{t_1} F_{x'} h'(t)dt = \int_{t_0}^{t_1} F_x h(t)dt - \int_{t_0}^{t_1} h(t)\frac{dF_{x'}}{dt}dt$$

Combine the integrals

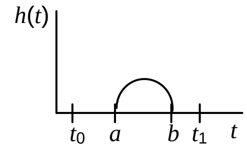
$$g'(a) \equiv \int_{t_0}^{t_1} \left(F_x - \frac{dF_{x'}}{dt}\right) h(t)dt$$

Trick 5 - this is Lemma 1 on p.16 of the text: Suppose that $z(t)$ is a given, continuous function defined on $[t_0, t_1]$. If

$$\int_{t_0}^{t_1} z(t)h(t)dt = 0$$

for every continuous function $h(t)$ defined on $[t_0, t_1]$ and satisfying $h(t_0) = h(t_1) = 0$, then $z(t) = 0$ for all $t \in [t_0, t_1]$

Note: this $h(t)$ is exactly the same as our $h(t)$; for this to apply we'll use $z(t) = F_x - dF_{x'}/dt$



Proof: (by contradiction)

Assume at some point between t_0 and t_1, $z(t) > 0$

Since $z(t)$ is continuous, there's a range (a,b) with $z(t) > 0$

The only condition for $h(t)$ is $h(t_0) = h(t_1) = 0$ so let's pick some $h(t)$ such that $h(t) > 0$ in the range (a,b)

That means the integral from a to b of $z(t)h(t)dt$ is > 0 which violates the given so there can't be a point where $z(t) > 0$ (similar argument for < 0)

Applying the Lemma - we know $g'(0) = 0$ so we have

$$g'(0) \equiv \int_{t_0}^{t_1} \left(F_x - \frac{dF_{x'}}{dt} \right) h(t)dt = 0$$

Note: since we're using an optimal solution now F represents $F(t, x^*(t), x^{*\prime}(t))$; the more complicated version goes away because $a = 0$

This satisfies the conditions of the Lemma if we use $z(t) = F_x - dF_{x'}/dt$ $\therefore$we have...

Euler's Equation - $\boxed{F_x = \frac{dF_{x'}}{dt} \text{ for } t_0 \le t \le t_1}$

Extremals - any solution to the Euler Equation; it's analogous to the first order condition $f'(x) = 0$ in static optimization (i.e., stationary points)

Trick 6 - the fun's not over, we can totally differentiate $F_{x'}$ wrt t; don't forget that $F_{x'}$ has three arguments: t, $x^*(t)$, $x^{*\prime}(t)$

$$\frac{dF_{x'}}{dt} = F_{x't}\frac{dt}{dt} + F_{x'x}\frac{dx}{dt} + F_{x'x'}\frac{dx'}{dt} = F_{x't} + F_{x'x}x' + F_{x'x'}x''$$

Now the Euler Equation is a "simple" second order differential equation:

$$F_x = F_{x't} + F_{x'x}x' + F_{x'x'}x''$$

Trick 7 - another way to write this is to integrate both sides; swamp them since the integral of $F_{x'}/dt = F_{x'}$:

duBois-Reymond Equation - $F_{x'}(t, x^*(t), x^{*\prime}(t)) = \int_0^t F_x(t, x^*(s), x^{*\prime}(s))\,ds + c$

Examples (Section I.4)

Production Example - from page 2 (p.22 in book):

$$\text{Min} \int_0^T \left[c_1[x'(t)]^2 + c_2 x(t) \right] dt \text{ s.t. } x'(t) \ge 0,\ x(0) = 0,\ x(T) = B$$

We'll use Euler Equation to find possible solutions

$$F(t, x(t), x'(t)) = c_1[x'(t)]^2 + c_2 x(t)$$

$$F_x(t, x(t), x'(t)) = c_2$$

$$F_{x'}(t, x(t), x'(t)) = 2c_1 x'(t)$$

$$\frac{dF_{x'}}{dt} = 2c_1 x''(t)$$

Euler Equation: $F_x = \frac{dF_{x'}}{dt} \Rightarrow c_2 = 2c_1 x''(t) \Rightarrow x''(t) = \frac{c_2}{2c_1}$

This is really a simple second order differential equation; integrate it wrt t:

$x'(t) = \int \frac{c_2}{2c_1} dt = \frac{c_2}{2c_1} t + k_1$ (where k_1 is an integration constant)

Integrate it wrt t again:

$x(t) = \int \left[\frac{c_2}{2c_1} t + k_1 \right] dt = \frac{c_2}{4c_1} t^2 + k_1 t + k_2$ (where k_2 is another integration constant)

Now we use the constraints $x(0) = 0$ and $x(T) = B$ to form 2 equations with 2 unknowns:

$x(0) = \frac{c_2}{4c_1}(0)^2 + k_1(0) + k_2 = k_2 \Rightarrow k_2 = 0$

$x(T) = \frac{c_2}{4c_1} T^2 + k_1 T + k_2 = B \Rightarrow k_1 = \frac{B}{T} - \frac{c_2}{4c_1} T$

We plug those back into $x(t)$ to get the form of the optimal solution:

$x(t) = \frac{c_2}{4c_1} t^2 + \left[\frac{B}{T} - \frac{c_2}{4c_1} T \right] t + 0 \Rightarrow \boxed{x(t) = \frac{c_2 t}{4c_1}(t - T) + \frac{Bt}{T}}$

Note: we ignored the constraint $x'(t) \geq 0$ so we have to go back and verify it

$x'(t) = \frac{c_2 t}{2c_1} - \frac{c_2}{4c_1} T + \frac{B}{T}$ ... don't help much

$x''(t) = \frac{c_2}{2c_1} > 0$ ... that means $x'(t)$ is strictly increasing iff $x'(t) \geq 0$ (think about it... x is convex because $x'' > 0$; if $x' \geq 0$, we're on the upward sloping side of the "U")

$x'(0) = \frac{c_2(0)}{2c_1} - \frac{c_2}{4c_1} T + \frac{B}{T} = -\frac{c_2}{4c_1} T + \frac{B}{T}$

In order for this to be ≥ 0, we must have $B \geq \frac{c_2}{4c_1} T^2$

Intuition - in order to guarantee this inequality we can look at several intuitive explanations: (1) the production quantity required (B) is very large compared to the time allowed (T); (2) inventory cost (c_2) is very small or production cost (c_1) is very big

Interpreting Euler Equation - $x''(t) = \frac{c_2}{2c_1} \Rightarrow 2c_1 x''(t) = c_2$

c_2 is the cost of holding one additional unit of inventory for one time period (i.e., marginal cost of inventory)

$c_1[x'(t)]^2$ is total production cost $\therefore$ $d\left(c_1[x'(t)]^2\right)/dt = 2c_1 x'(t)$ is the instantaneous marginal cost of production and $d\left(2c_1 x'(t)\right)/dt = 2c_1 x''(t)$ is the time rate of change of the marginal cost of production $\therefore$ Euler equation calls for balancing the rate of change of the marginal production cost against the marginal inventory holding cost

Another way to look at it is to integrate the Euler equation over a small time interval Δ

$\int_t^{t+\Delta} 2c_1 x''(s) ds = \int_t^{t+\Delta} c_2 ds \Rightarrow 2c_1 x'(s)\Big]_t^{t+\Delta} = c_2 s\Big]_t^{t+\Delta} \Rightarrow$

$2c_1\left[x'(t+\Delta) - x'(t)\right] = c_2\left[t + \Delta - t\right] = c_2 \Delta$

Move $-x'(t)$ term to right side: $2c_1 x'(t+\Delta) = c_2 \Delta + 2c_1 x'(t)$

This says that the marginal cost to produce at time t + Δ is the same as the marginal cost to produce now (time t) and store it until time t + Δ which implies this is optimal because we can't shift the production schedule to reduce cost

Dynamic Consumption-Savings Example - p.26 in book; individual seeks consumption rate at each moment of time that will maximize his discounted utility stream over a period of known length T; utility of consumption at each moment t is $U(C(t))$ which is increasing concave function (diminishing marginal utility: $U' > 0$ and $U'' < 0$); future utility is discounted at rate r

$$\int_0^T e^{-rt} U(c(t))dt$$

subject to cash flow constraint. individual derives current income form exogenously determined wages $w(t)$ and from interest earnings i on his holdings of capital assets $K(t)$. For simplicity, the individual may borrow capital ($K < 0$) as well as lend it at interest rate i. Capital can be bought or sold at a price of unity. Thus income from interest and wages is allotted to consumption and investment:

$$iK(t) + w(t) = C(t) + K'(t)$$

Initial and terminal capital stocks are specified: $K(0) = K_0$ and $K(T) = K_T$

We'll use Euler Equation to find possible solutions

First, note that $C(t) = iK(t) + w(t) - K'(t)$ we our objective function actually is:

$$\int_0^T e^{-rt} U(iK(t) + w(t) - K'(t))dt$$

Now we can use the Euler Equation:

$F(t, K(t), K'(t)) = e^{-rt} U(iK(t) + w(t) - K'(t))$

$F_K(t, K(t), K'(t)) = e^{-rt} U'(C(t))i$ ... e^{-rt} is constant wrt $K(t)$; uses the chain rule on $U(C(t))$

$F_{K'}(t, K(t), K'(t)) = -e^{-rt} U'(C(t))$

Euler Equation: $F_x = \dfrac{dF_{x'}}{dt} \;\Rightarrow\; e^{-rt} U'(C(t))i = \dfrac{d}{dt}\left(-e^{-rt} U'(C(t))\right)$

What Does it Mean - use same trick of integrating over a small period of time Δ:

$$\int_t^{t+\Delta} e^{-rs} U'(C(s))i ds = \int_t^{t+\Delta} \frac{d}{ds}\left(-e^{-rs} U'(C(s))\right)ds$$

Workout integral on right side:

$$\int_t^{t+\Delta} e^{-rs} U'(C(s))i ds = -e^{-rs} U'(C(s))\Big]_t^{t+\Delta} = -e^{-r(t+\Delta)} U'(C(t+\Delta)) + e^{-rt} U'(C(t))$$

Move $e^{-rt}U'(C(t))$ term to right side by itself:

$$\underbrace{e^{-rt} U'(C(t))}_{\text{marginal discounted utility from consumption at t}} = \underbrace{\left(\int_t^{t+\Delta} e^{-rs} U'(C(s))i ds\right) + e^{-r(t+\Delta)} U'(C(t+\Delta))}_{\text{marginal discounted utility from postponing consumption until t + }\Delta}$$

assumes postponed dollar earns income at rate i that is consumed as earned (first term), then dollar is consumed at $t + \Delta$

We still have to solve for C(t) so let's go back to Euler Equation and work out the right side:

$$\frac{d}{dt}\left(-e^{-rt}U'(C(t))\right) = re^{-rt}U'(C(t)) - e^{-rt}U''(C(t))C'(t)$$

Put that back into Euler Equation: $re^{-rt}U'(C(t)) - e^{-rt}U''(C(t))C'(t) = e^{-rt}U'(C(t))i$

Cancel out e^{-rt} and drop the function arguments: $rU' - U''C' = U'i$

Solve for i - r: $\frac{-U''C'}{U'} = i - r$

This is a second order differential equation; note that -U''/U' is absolute risk aversion, which for this problem is > 0 because of the assumptions made about U; ∴ if $i > r$, we will have C' > 0 (person will invest and consumption will increase over time); if $i < r$, we will have C' < 0 (person will consume more now and consumption will decrease over time)

Specific $U(C(t))$ - assume $U(C(t)) = \ln C$, with $w(t) = 0$ for $0 \le t \le T$ (individual doesn't work), and $K(T) = 0$ (consume all assets by time T)

Solve for U' and U'':

$$U' = \frac{1}{C},\ U'' = -\frac{1}{C^2},\ \therefore\ -\frac{U''}{U'} = -\frac{-1/C^2}{1/C} = \frac{1}{C}$$

Plug that into the general solution we found before:

$\frac{-C'}{C} = i - r$ ... which simplifies to $C'(t) - (i-r)C(t) = 0$

Our second order differential equation is now a first order differential equation; we can use DiffEq Trick #3 (see separate notes)

Multiply both sides by $e^{-(i-r)t}$:

$$e^{-(i-r)t}C'(t) - (i-r)e^{-(i-r)t}C(t) = 0$$

Note that the left side simplifies (it's part of trick #3):

$$e^{-(i-r)t}C'(t) - (i-r)e^{-(i-r)t}C(t) = \frac{d}{dt}\left(e^{-(i-r)t}C(t)\right) = 0$$

Integrate both sides:

$$\int_0^t \frac{d}{ds}\left(e^{-(i-r)s}C(s)\right)ds = 0$$

Work out the integral:

$$e^{-(i-r)s}C(s)\Big]_o^t = e^{-(i-r)t}C(t) - e^{-(i-r)0}C(0) = 0$$

Solve for $C(t)$:

$$C(t) = e^{(i-r)t}C(0)$$

Now we have an equation for $C(t)$, but we don't know what $C(0)$ is... gotta keep working

Use the cash flow constraint: $iK(t) + w(t) = C(t) + K'(t)$

Rearrange terms and substitute for $C(t)$:

$$K'(t) - iK(t) = -e^{(i-r)t}C(0)$$

We have another first order differential equation that uses DiffEq Trick #3; multiply both sides by e^{-it}:

$$e^{-it}K'(t) - e^{-it}iK(t) = -e^{(i-r)t}C(0)e^{-it} = -e^{-rt}C(0)$$

The left side simplifies to a derivative (part of trick #3):

$$e^{-it}K'(t) - e^{-it}iK(t) = \frac{d}{dt}\left(e^{-it}K(t)\right)$$

Since we're going to integrate both sides, we have to try to apply the same trick to the right side:

$$-e^{-rt}C(0) = \frac{d}{dt}\left(\frac{1}{r}e^{-rt}C(0) + \ell\right)$$, where ℓ is a constant

Put these together:

$$\frac{d}{dt}\left(e^{-it}K(t)\right) = \frac{d}{dt}\left(\frac{1}{r}e^{-rt}C(0) + \ell\right)$$

Now integrate both sides:

$$\int \frac{d}{dt}\left(e^{-it}K(t)\right)dt = \int \frac{d}{dt}\left(\frac{1}{r}e^{-rt}C(0) + \ell\right)dt$$

This is the easy step:

$$e^{-it}K(t) = \frac{1}{r}e^{-rt}C(0) + \ell$$

Now solve for $K(t)$:

$$K(t) = \frac{1}{r}e^{(i-r)t}C(0) + \ell e^{it}$$

Now we have an equation for $K(t)$ which we can combine with $C(t)$ to have two equations with two unknowns: $C(0)$ and ℓ

Use $K(0) = K_0$ and $K(T) = 0$:

$$K(0) = K_0 = \frac{1}{r}e^{(i-r)0}C(0) + \ell e^{i0} = \frac{1}{r}C(0) + \ell \;\Rightarrow\; \ell = K_0 - \frac{1}{r}C(0)$$

$$K(T) = 0 = \frac{1}{r}e^{(i-r)T}C(0) + \ell e^{iT}$$

Substitute ℓ from the $K(0)$ equation:

$$0 = \frac{1}{r}e^{(i-r)T}C(0) + \left(K_0 - \frac{1}{r}C(0)\right)e^{iT} = \frac{1}{r}e^{(i-r)T}C(0) + K_0 e^{iT} - \frac{1}{r}C(0)e^{iT}$$

Factor out the $C(0)e^{it}/r$:

$$0 = \frac{1}{r}C(0)e^{it}\left(e^{-rT} - 1\right) + K_0 e^{iT}$$

Solve for $C(0)$:

$$C(0) = \frac{-rK_0}{e^{-rT} - 1}$$

Plug back into equation for ℓ:

$$\ell = K_0 - \frac{1}{r}\left(\frac{-rK_0}{e^{-rT} - 1}\right) = K_0 + \frac{K_0}{e^{-rT} - 1} = K_0\left(1 + \frac{1}{e^{-rT} - 1}\right)$$

Plug these back into the equations for $C(t)$ and $K(t)$:

$$C(t) = \frac{-rK_0 e^{(i-r)t}}{e^{-rT} - 1}$$ ... just an easy substitution

$$K(t) = \frac{1}{r}e^{(i-r)t}\left(\frac{-rK_0}{e^{-rT} - 1}\right) + K_0\left(1 + \frac{1}{e^{-rT} - 1}\right)e^{it}$$ ... need to simplify

Multiply out first term and move e^{it} to front of second term:

$$K(t) = \frac{-K_0 e^{(i-r)t}}{e^{-rT}-1} + e^{it}K_0\left(1+\frac{1}{e^{-rT}-1}\right)$$

Factor out $e^{it}K_0$ from first term:

$$K(t) = e^{it}K_0\left(\frac{-e^{-rt}}{e^{-rT}-1}\right) + e^{it}K_0\left(1+\frac{1}{e^{-rT}-1}\right)$$

Now factor out $e^{it}K_0$ from both terms:

$$K(t) = e^{it}K_0\left(1+\frac{1-e^{-rt}}{e^{-rT}-1}\right)$$

Don't know why, but make the faction look the same by multiplying the denominator by -1:

$$K(t) = e^{it}K_0\left(1-\frac{1-e^{-rt}}{1-e^{-rT}}\right)$$

Final solution:

$$C(t) = \frac{-rK_0 e^{(i-r)t}}{e^{-rT}-1} \text{ and } K(t) = e^{it}K_0\left(1-\frac{1-e^{-rt}}{1-e^{-rT}}\right)$$

Free End Value (Section I.8)

Now generalize the simplest continuous, dynamic optimization problem by removing the ending condition (i.e., $x(t_1)$ is free to be anything):

$$\text{Max } \int_{t_0}^{t_1} F(t, x(t), x'(t))dt \text{ s.t. } x(t_0) = x_0$$

Example - using the production example we were using before:

$$\text{Min } \int_0^T e^{-rt}\left[c_1[x'(t)]^2 + c_2 x(t)\right]dt \text{ s.t. } x'(t) \geq 0,\ x(0) = 0,\ x(T) = B$$

Since we had a fixed total output B, we did cost minimization rather than profit maximization (they give the same answer in this case); since we want to let B vary in order to maximize profit, we need to look at the profit maximization problem:

$$\text{Max } \boldsymbol{\pi} = Be^{-rT} - \int_0^T e^{-rt}\left[c_1[x'(t)]^2 + c_2 x(t)\right]dt \text{ s.t. } x'(t) \geq 0,\ x(0) = 0$$

Total Production - $\int_0^T x'(t)dt = x(T) \equiv B$ (is now variable); using the integral form of production, the objective becomes:

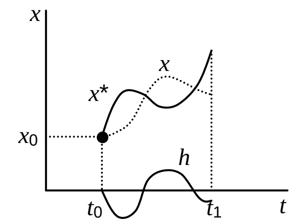
$$\text{Max } \boldsymbol{\pi} = \int_0^T e^{-rt}\left[x'(t) - c_1[x'(t)]^2 - c_2 x(t)\right]dt \text{ s.t. } x'(t) \geq 0,\ x(0) = 0$$

In order to figure out how to solve this, we need to do lots of tricks like we did with the simple version before (when we derived the Euler Equation); we'll work with the general form listed above (integrating from t_0 to t_1)

Admissible Function - "feasible" solution to optimization problem; a function $x(t)$ that is continuously differentiable defined on the interval $[t_0,t_1]$ satisfying the fixed endpoint condition ($x(t_0) = x_0$)

Optimal Solution - assume $x^*(t)$ solves the problem

Difference Function - assume $x(t)$ is some other admissible (feasible) function; the difference function is $h(t) = x(t) - x^*(t)$;



Note: we picked $x(t)$ (and hence $h(t)$) arbitrarily, but once selected they are fixed; also, note that $h(t_0) = 0$ because both $x(t)$ and $x^*(t)$ are admissible so they have the same value at t_0

General Admissible Function - $g(t) = x^*(t) + ah(t)$ must be admissible for all a because it is continuously differentiable and satisfies the fixed endpoint condition: $g(t_0) = x^*(t_0) + ah(t_0) = x_0 + a(0) = x_0$

General is Optimal - since $x^*(t)$ is optimal, $g(t)$ will be optimal if $a = 0$

General as Function of *a* - since $x^*(t)$ and $h(t)$ are fixed, we can look at g as a function of a... $g(a)$; we just said g is optimal if $a = 0$; we can combine that with the first order condition to get $g'(0) = 0$

Put it Back - now plug $g(t)$ into original objective: $g(a) \equiv \int_{t_0}^{t_1} F(t, g(t), g'(t))dt$

Substitute $g(t) = x^*(t) + ah(t)$ and $g'(t) = x^{*\prime}(t) + ah'(t)$:

$$g(a) \equiv \int_{t_0}^{t_1} F(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))dt$$

Trick 1 - we want to work with $g'(0)$ instead of $g(a)$ we need to differentiate the identity above with respect to a and evaluate it at $a = 0$:

Note: in order to save space, $F(t,x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))$ will just be written as F

$$g'(a) \equiv \int_{t_0}^{t_1} \frac{dF}{da}dt$$

Trick 2 - we are totally differentiating F by a:

$$\frac{dF}{da} = F_t \frac{dt}{da} + F_x \frac{d(x^*(t) + ah(t))}{da} + F_{x'} \frac{d(x^{*\prime}(t) + ah'(t))}{da} = F_x h(t) + F_{x'} h'(t)$$

Note: F_i is the partial derivative of F with wrt the i^{th} argument ($i = t, x, x'$)

Note: $dt/da = 0$

Now we can rewrite g'(a):

$$g'(a) \equiv \int_{t_0}^{t_1} [F_x h(t) + F_{x'} h'(t)]dt$$

Trick 3 - we want to get the two terms to have the same $h(t)$ term so we need to get ride of the $h'(t)$ in the second term; we'll do that by breaking up the integral, solving the second part and then recombining:

$$g'(a) \equiv \int_{t_0}^{t_1} F_x h(t)dt + \int_{t_0}^{t_1} F_{x'} h'(t)dt$$

Trick 4 - use integration by parts in order to evaluate the second integral

$$\int udv = uv - \int vdu$$

To apply that to the second term in $g'(a)$, we'll use:

$$u = F_x' \text{ and } dv = h'(t)dt = \frac{dh(t)}{dt}dt = dh(t) \Rightarrow du = \frac{dF_x'}{dt} \text{ and } v = h(t)$$

Now using integration by parts:

$$\int_{t_0}^{t_1} F_{x'} h'(t)dt = F_{x'} h(t)\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} h(t)\frac{dF_{x'}}{dt}dt = F_{x'}h\Big|_{t_1} - \int_{t_0}^{t_1} h(t)\frac{dF_{x'}}{dt}dt$$

Note: the $Fx'h(t_0)$ term goes away because $h(t_0) = 0$

Notation - $F_{x'}h\Big|_{t_1}$ means we evaluate at t_1 (i.e., $F_{x'}(t_1,x(t_1),x'(t_1))h(t_1)$)

Now add that back into $g'(a)$

$$g'(a) \equiv \int_{t_0}^{t_1} F_x h(t)dt + \int_{t_0}^{t_1} F_{x'} h'(t)dt = \int_{t_0}^{t_1} F_x h(t)dt + F_{x'}h\Big|_{t_1} - \int_{t_0}^{t_1} h(t)\frac{dF_{x'}}{dt}dt$$

Combine the integrals

$$g'(a) \equiv F_{x'}h\Big|_{t_1} + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)h(t)dt$$

Trick 5 - now let $a = 0$ so we have

$$g'(0) \equiv F_{x'}h\Big|_{t_1} + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)h(t)dt = 0$$

Note: since we're using an optimal solution now F represents $F(t, x^*(t), x^{*\prime}(t))$; the more complicated version goes away because $a = 0$

Since we were allowed to pick any feasible function $x(t)$, we could've picked with the same endpoint as $x^*(t)$ so we could have $h(t_1) = 0$; that would remove the first term and require

$$\int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)h(t)dt = 0$$

Based on lemma 1 from p.16 (same one we used in step 5 before), this leads us to the Euler Equation:

$$\boxed{F_x = \frac{dF_{x'}}{dt} \quad \text{for } t_0 \le t \le t_1}$$

Trick 6 - we're not done because $h(t_1)$ doesn't have to be zero. We did learn, however that the second term of $g'(0)$ is zero so the first term must also be zero... and since $h(t_1)$ isn't necessarily zero, we have a new condition to describe an **extremal** (equivalent to stationary points in static optimization):

Transversality Condition - $\boxed{F_{x'}(t_1, x(t_1), x'(t_1)) = 0}$

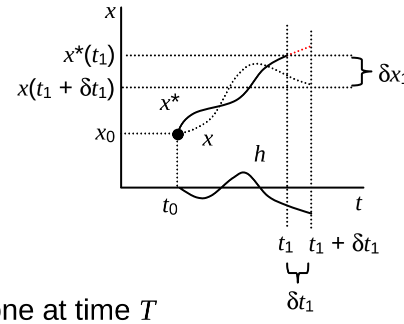
Free Horizon (Section I.9)

Hotelling Monopoly Problem - 1931 paper; how to extract a resource over time to maximize present value of profits $\pi(q(t))$ subject to the resource constraint:

$$\int_0^\infty q(t)dt \le B$$

where $q(t)$ is amount extracted at time t and B is total stock of resource available

Cumulative Extraction - $x(t) = \int_0^t q(s)ds$



Instantaneous Extraction - $x'(t) = q(t)$

Problem - maximize present value of profit:

$$\text{Max} \int_0^{tT} e^{-rt}\pi(x'(t))dt \ \text{ s.t. } x(0) = 0,\ x(T) = B$$

End Conditions - start with no extraction; end with everything gone at time T

Solution - this is a version of the simplest problem (Euler equation only)

$$F(t, x(t), x'(t)) = e^{-rt}\boldsymbol{\pi}(x'(t))$$

$$F_x = 0$$

$F_{x'} = e^{-rt}\dfrac{\partial\boldsymbol{\pi}(x'(t))}{\partial x'(t)}$ ... note that $\dfrac{\partial\boldsymbol{\pi}(x'(t))}{\partial x'(t)}$ is marginal profit

Euler Equation - $F_x = \dfrac{d}{dt}F_{x'} \ \Rightarrow\ 0 = \dfrac{d}{dt}\left(e^{-rt}\dfrac{\partial\boldsymbol{\pi}(x'(t))}{\partial x'(t)}\right)$

Integrate both sides: $F_{x'} = c$ (some constant); i.e. marginal profit of extraction is constant over time (so given 1 extra unit of resource, the monopolist doesn't care which period it gets extracted because it yields the same amount of profit in each period)

Problem? - why did we say all the resources had to be extracted by time T? If they don't, we have to use a free horizon

Generalize the simplest continuous, dynamic optimization problem by removing the ending period (i.e., t_1 is now variable)

$$\text{Max} \int_{t_0}^{t_1} F(t, x(t), x'(t))dt \ \text{ s.t. } x(t_0) = x_0$$

Optimal Solution - assume $x^*(t)$ is defined over $[t_0,t_1]$ and solves the problem

Admissible Function - "feasible" solution $x(t)$ which is continuously differentiable defined on the interval $[t_0,t_1 + \delta t_1]$ satisfying the endpoint condition ($x(t_0) = x_0$)

Extending the Domain - basic idea is to take the function with the smaller domain and use linear extrapolation to cover the larger domain; by linear extrapolation, we take the slope of the function at it's end point and extend the line out to the other function's endpoint (in the domain)

$\delta t_1 > 0$ - stretch $x^*(t)$: $x^*(t) = x^*(t_1) + x^{*\prime}(t_1)(t - t_1)$, $t_1 \le t \le t_1 + \delta t_1$

$\delta t_1 < 0$ - stretch $x(t)$: $x(t) = x(t_1) + x'(t_1)(t - t_1)$, $t_1 + \delta t_1 \le t \le t_1$

Difference Function - $h(t) = x(t) - x^*(t)$; defined over $t_0 \le t \le \text{Max}(t_1, t_1 + \delta t_1)$

General Admissible Function - $g(t) = x^*(t) + ah(t)$ defined over $[t_0, t_1 + a\delta t_1]$ must be admissible for all a because it is continuously differentiable and satisfies the fixed endpoint condition: $g(t_0) = x^*(t_0) + ah(t_0) = x_0 + a(0) = x_0$

General is Optimal - as before since $x^*(t)$ is optimal, $g(t)$ will be optimal if $a = 0$

General as Function of *a* - since $x^*(t)$ and $h(t)$ are fixed, we can look at g as a function of a... $g(a)$; we just said g is optimal if $a = 0$; we can combine that with the first order condition to get $g'(0) = 0$; looking at $g(a)$ now we have:

$$g(a) \equiv \int_{t_0}^{t_1 + a\delta t_1} F(t, x^*(t) + ah(t), x^{*\prime}(t) + ah'(t))dt$$

Trick 1 - we want to work with $g'(0)$ instead of $g(a)$ we need to differentiate the identity above with respect to a and evaluate it at $a = 0$

Leibnitz's Rule - $\frac{d}{da}\int_a^{b(t)} f(x,t)dx = \int_a^{b(t)} f_t(x,t)dx + f(b(t),t)\frac{db}{dt}$ (note there are only 2 terms instead of 3 because a is not a function of t); applying this to g(a), we get:

$$g'(0) = F(t_1, x^*(t_1), x^{*\prime}(t_1))\delta t_1 + \int_{t_0}^{t_1}(F_x h + F_{x'}h')dt = 0$$

Trick 2 - use integration by parts to evaluate the second integral (just like tricks 2-4 in last section):

$$g'(0) = F(t_1, x^*(t_1), x^{*\prime}(t_1))\delta t_1 + F_{x'}(t_1, x^*(t_1), x^{*\prime}(t_1))h(t_1) + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)hdt = 0$$

Or using the "evaluate at" notation (purposely not including $h(t_1)$ because we're going to substitute for it in the next trick):

$$g'(0) = F\big|_{t_1}\delta t_1 + F_{x'}\big|_{t_1} h(t_1) + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)hdt = 0$$

Trick 3 - call the difference in the value of x(t) and x*(t):

$\delta x_1 \equiv x(t_1 + \delta t_1) - x^*(t_1)$

Now approximate $x(t_1 + \delta t_1) \approx x(t_1) + x^{*\prime}(t_1)\delta t_1$

Substitute back so $\delta x_1 \approx x(t_1) + x^{*\prime}(t_1)\delta t_1 - x^*(t_1)$

Note that $h(t_1) = x(t_1) - x^*(t_1)$ so we have $\delta x_1 \approx h(t_1) + x^{*\prime}(t_1)\delta t_1$

This means that the difference in value of the functions at their respective terminal points is approximated by their difference in value at t_1 plus the change in value over the interval between t_1 and $t_1 + \delta t_1$

Solve for $h(t_1)$: $h(t_1) \approx \delta x_1 - x^{*\prime}(t_1)\delta t_1$

Now substitute this into $g'(0)$:

$$g'(0) = F\big|_{t_1}\delta t_1 + F_{x'}\big|_{t_1}\left[\delta x_1 - x^{*\prime}(t_1)\delta t_1\right] + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)hdt = 0$$

Combine terms with δt_1:

$$g'(0) = (F - x'F_{x'})\big|_{t_1}\delta t_1 + F_{x'}\big|_{t_1}\delta x_1 + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)hdt = 0$$

Note: the book uses x' in the first term, but Len still thinks it's supposed to be $x^{*\prime}$ because the slopes of those functions are not necessarily the same at t_1

Trick 4 - Since x could terminate at exactly the same point that x^* does, we'd have $\delta t_1 = \delta x_1 = 0$, which means we must have

$$\int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)h(t)dt = 0$$

Based on lemma 1 from p.16 (same one we used in step 5 before), this leads us to the Euler Equation:

$$\boxed{F_x = \frac{dF_{x'}}{dt} \quad \text{for } t_0 \le t \le t_1}$$

Trick 5 - **Transversality Conditions**; Now that we can eliminate the integral term in $g'(0)$, we have:

$$g'(0) = \left(F - x'F_{x'}\right)\Big|_{t_1} \delta t_1 + F_{x'}\Big|_{t_1} \delta x_1 = 0$$

We can look at three cases:

(1) t_1 fixed (i.e., $\delta t_1 = 0$) and $x(t_1)$ free; that means

$$\boxed{F_{x'}(t_1, x(t_1), x'(t_1)) = 0}$$

Intuition - looks like first order condition; marginal benefit of changing x' is zero

(2) t_1 free and $x(t_1)$ fixed (i.e., $\delta x_1 = 0$); that means

$$\boxed{\left(F - x'F_{x'}\right)\Big|_{t_1} = 0}$$

Intuition - $x(t_1)$ fixed, but if we change t_1 then $x(t_1)$ changes unless we change x'; the effect of that is $x'F_{x'}$ evaluated at t_1; the marginal benefit of changing t_1 would be:

$$\frac{d}{dt}\int_{t_0}^{t_1} F(t, x, x')dt\Bigg|_{t_1} = F\Big|_{t_1}$$

so intuition of this condition is marginal benefit of changing t_1 equal the marginal cost of changing x' (due to changing t_1)

(3) Both t_1 and $x(t_1)$ free; has to work for both $t_1 = 0$ and $x(t_1) = 0$, but not necessarily at the same time; that means we have both conditions 1 and 2, which lets us simplify because we can substitute (1) into (2):

$$\boxed{F\Big|_{t_1} = F_{x'}\Big|_{t_1} = 0}$$

Legendre Conditions - (covered in I.6) necessary conditions for max and min problems:

Max - $F_{x'x'} \leq 0$ for $t_0 \leq t \leq t_1$

Min - $F_{x'x'} \geq 0$ for $t_0 \leq t \leq t_1$

Exercise 1 - p.63 in book; Let B be the total quantity of some exhaustible resource, for example, the amount of mineral in a mine, controlled by a monopolist who discounts continuously at rate r and wishes to maximize the present value of profits from the mine. Let $x(t)$ be the cumulative amount sold by time t and $x'(t)$ be the current rate of sales. The net price (gross price less cost of mining) $p(x')$ is assumed to be a decreasing, continuously differentiable function of the current rate of sales:

$$p'(x') < 0$$

(a) Let T denote the time at which the resource will be depleted. Then choose $x(t)$ and T to maximize

$$\int_0^T e^{-rt} p(x'(t))x'(t)dt \text{ subject to } x(0) = 0 \text{ and } x(T) = B$$

Employ the Euler equation, transversality condition, and Legendre condition to show that the optimal plan involves sales decreasing over time, with $x'(T) = 0$

(b) Show that at T, the average profit per unit of resource extraction just equals the marginal profit.

(c) Find the solution if

$$p(x') = \left(1 - e^{-kx'}\right)/\,x'$$

where $k > 0$ is a given constant. (Partial answer: $T = (2kB/r)^{1/2}$.)
Note that increasing the initial stock by a fraction lengthens the extraction period only by a smaller fraction; that is, $T^*(B)$ is an increasing concave function.

$$F(t, x(t), x'(t)) = e^{-rt} p(x'(t)) x'(t)$$

$$F_x = 0$$

$$F_{x'} = e^{-rt} p'(x'(t)) x'(t) + e^{-rt} p(x'(t))$$

Euler Equation - $F_x = \dfrac{dF_{x'}}{dt} \;\Rightarrow\; 0 = \dfrac{d}{dt}\left[e^{-rt}\left(p'(x'(t))x'(t) + p(x'(t))\right)\right]$

Integrate both sides and we get: $c_1 = F_{x'} = e^{-rt}\left[p'(x'(t))x'(t) + p(x'(t))\right]$ = marginal profit at time t (it's derivative of profit wrt x' [sales rate; amount of resource sold at time t])

Transversality - problem has t_1 free (get to choose T) and $x(T)$ fixed at B $\therefore$

$\left(F - x'F_{x'}\right)\Big|_{t_1} = 0 = e^{-rt}\left[p(x'(T))x'(T) - p'(x'(T))x'(T) - p(x'(T))\right] = e^{-rt} p'(x'(T))x'(T)$;

we're given that $p'(x') < 0$ so we must have $\boldsymbol{x'(T) = 0}$ which is the first thing we're supposed to show (don't have any output at time T; "no output in last period" is discrete time equivalent)

Substitute back into Euler Equation (which holds for all $t \in [t_0, t_1]$, including $t_1 = T$)

$$c_1 = e^{-rt}\left[p'(x'(T))x'(T) + p(x'(T))\right] = e^{-rt}\left[p'(0)\cdot 0 + p(0)\right] = e^{-rt} p(0)$$

For any reasonable demand curve, we must have $p(0) > 0 \;\Rightarrow\; c_1 > 0$

We want to show $x'(t)$ decreasing over time (i.e., $dx'(t)/dt = x''(t) < 0$)... take total derivative of Euler equation wrt t:

$$\frac{dc_1 e^{rt}}{dt} = c_1 r e^{rt} = \frac{d}{dt}\left(p'(x'(t))x'(t) + p(x'(t))\right) =$$

$$p''(x'(t))x'(t)x''(t) + p'(x'(t))x''(t) + p'(x'(t))x''(t) = x''(t)\left(p''(x'(t) + 2p'(x'(t))\right)$$

$$x''(t) = \frac{c_1 r}{e^{-rt}\left[p''(x'(t)) + 2p'(x'(t))\right]}$$

The numerator is positive because we showed that $c_1 > 0$ (and know r and $e^{-rt} > 0$)

Legendre Condition - this is a max problem so $F_{x'x'} \le 0$

$$\frac{\partial F_{x'}}{\partial x'} = e^{-rt}\left[p''(x'(t))x'(t) + p'(x'(t)) + p'(x'(t))\right] = e^{-rt}\left[p''(x'(t))x'(t) + 2p'(x'(t))\right]$$

which is the denominator of $x''(t)$ $\therefore$ $x''(t) < 0$ so we know $\boldsymbol{x'(t)}$ **is decreasing**

Don't think we covered part b in class

Look at $p(x') = \left(1 - e^{-kx'}\right)/\,x'$

$$F(t, x(t), x'(t)) = e^{-rt}\frac{\left(1 - e^{-kx'}\right)}{x'}x' = e^{-rt}\left(1 - e^{-kx'}\right)$$

$$F_{x'} = ke^{-rt}e^{-kx'}$$

Euler Equation - $c_1 = F_{x'} = ke^{-rt}e^{-kx'}$

We want to solve for x(t), so take $\ln$ of both sides:

$$\ln c_1 = \ln k - rt - kx' \;\Rightarrow\; kx' = -\ln c_1 + \ln k - rt \;\Rightarrow\; x' = \frac{\ln(k/c_1) - rt}{k}$$

$x'(T) = 0 \Rightarrow \ln(k/c_1) - rT = 0 \;\Rightarrow\; \ln(k/c_1) = rT$

Substitute that back into the Euler equation: $x' = \dfrac{rT - rt}{k}$

We know $x(T) = \int_0^T x'(t)dt = B$ so we can use that to solve for T:

$$B = \int_0^T x'(t)dt = \int_0^T \frac{rT - rt}{k}dt = \frac{rT}{k}t\Big]_0^T - \left(\frac{rt^2}{2k}\Big]_0^T\right) = \frac{rT^2}{k} - \frac{rT^2}{2k} = \frac{rT^2}{2k}$$

Solving to T: $T = \sqrt{\dfrac{2kB}{r}}$ ... this is where we realize we did a bunch of math to find that $B\uparrow \Rightarrow T\uparrow$ (i.e., take longer with more resource) and $r\uparrow \Rightarrow T\downarrow$ (i.e., use resource quicker if more impatient)... at least the answer makes sense!

Plug that back into Euler equation: $x' = \dfrac{r\left(\frac{2kB}{r}\right)^{1/2} - rt}{k} = \left(\dfrac{2Br}{k}\right)^{1/2} - \dfrac{rt}{k}$

Now integrate that to solve for x(t):

$$x(t) = \int_0^t x'(s)ds = \int_0^t \left[\left(\frac{2Br}{k}\right)^{1/2} - \frac{rs}{k}\right]ds = \left(\frac{2Br}{k}\right)^{1/2} t - \frac{rt^2}{2k} + c_2$$

Initial Condition: $x(0) = 0 \;\Rightarrow\; c_2 = 0$

Solution... finally... is

$$x(t) = \left(\frac{2Br}{k}\right)^{1/2} t - \frac{rt^2}{2k}$$

Summary

Basic Problem:

$$\text{Max or Min } \int_{t_0}^{t_1} F(t, x(t), x'(t))dt$$

$$\text{subject to } x(t_0) = x_0$$

Optional Constraints - pick neither, one or both; if we use both, we don't need the transversality conditions

***x*(*t*$_1$) Fixed** - set a definite value for x in the last period; $x(t_1) = x_1$

***t*$_1$ Fixed** - set a definite end time; $t_1 = T$

Euler Equation - $F_x = \dfrac{dF_{x'}}{dt}$ for $t_0 \le t \le t_1$

Legendre Conditions -

Max - $F_{x'x'} \le 0$ for $t_0 \le t \le t_1$

Min - $F_{x'x'} \ge 0$ for $t_0 \le t \le t_1$

Transversality Conditions - these are evaluated at t_1 (not all t like the other conditions)

(1) $x(t_1)$ free: $F_{x'}\big|_{t_1} = 0$

(2) t_1 free: $\left(F - x'F_{x'}\right)\big|_{t_1} = 0$

(3) Both t_1 and $x(t_1)$ free: $F\big|_{t_1} = F_{x'}\big|_{t_1} = 0$

Diagrammatic Analysis

Consider the problem

$$\text{Min} \int_0^T e^{-rt}\left[f(x'(t)) + g(x(t))\right]dt \text{ s.t. } x(0) = x_0 \text{ and } x(T) = x_T$$

where $f'' > 0$ and $g'' > 0$. The functions f and g are twice continuously differentiable and strictly convex, but not further specified

Euler Equation -

$$F(t, x(t), x'(t)) = e^{-rt}\left[f(x'(t)) + g(x(t))\right]$$

$$F_x = e^{-rt} g'(x(t))$$

$$F_{x'} = e^{-rt} f'(x'(t))$$

$$\frac{dF_{x'}}{dt} = e^{-rt} f''(x'(t))x''(t) - re^{-rt} f'(x'(t)) = e^{-rt}\left[f''(x')x'' - rf'(x')\right]$$

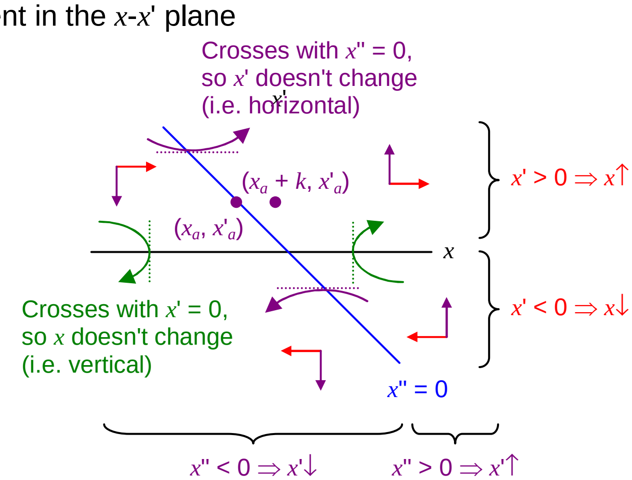
Euler Equation: $F_x = \dfrac{dF_{x'}}{dt} \;\Rightarrow\; e^{-rt} g'(x(t)) = e^{-rt}\left[f''(x')x'' - rf'(x')\right]$

e^{-rt} cancels; solve for x'': $\boxed{x'' = \dfrac{rf'(x') + g'(x)}{f''(x')}}$

Can't Do More - without further specification of f and g we can't solve further

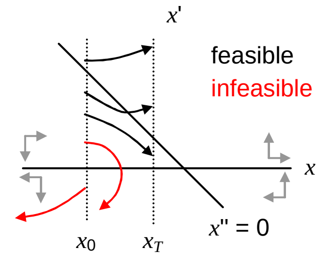
Autonomous Differential Equation - t is not an argument to the Euler equation we found

Diagrammatic Analysis - look at directions of movement in the x-x' plane



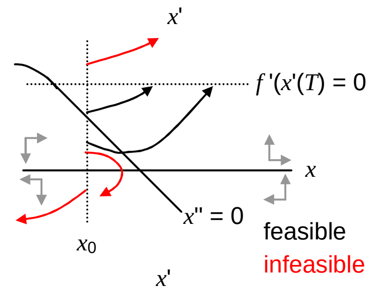
1. **First Derivative** - $x' > 0 \Rightarrow x$ increasing; $x' < 0 \Rightarrow x$ decreasing; $\therefore$ we can draw horizontal line at $x' = 0$; above that line $x\uparrow$ (moves right), below that line $x\downarrow$ (moves left)
2. **Crossing $x' = 0$** - only time x doesn't change (i.e., vertical line) is at $x' = 0$ (on the "horizontal axis")
3. **$x'' = 0$** - from Euler equation this implies $rf'(x') + g'(x) = 0$; totally differentiate this: $rf''(x')dx' + g''(x)dx = 0 \;\Rightarrow$ $\dfrac{dx'}{dx} = -\dfrac{g''(x)}{rf''(x')} < 0$ since we're given that $f'' > 0$ and $g'' > 0$. (r is a discount rate so it is assumed to be > 0 as well.) $\therefore$ as we increase x, x' is decreasing (i.e., line determined by $x'' = 0$ in the x-x' plane has negative slope)
4. **Crossing $x'' = 0$** - pick point on $x'' = 0$, say (x_a, x'_a) with $x'_a > 0$; now add some positive amount k to x_a and use Euler equation to determine sign of x''; (since denominator is > 0, we can ignore it)
   $rf'(x'_a) + g'(x_a) = 0$ and $rf'(x'_a) + g'(x_a + k) > 0$, because $g'' > 0$ (i.e., $x\uparrow \Rightarrow g'(x)\uparrow$) $\therefore$ $x'' > 0$ to the right of $x'' = 0$; which means x' increases; similarly, left of $x'' = 0$, we have x' decreasing $\therefore$ at $x'' = 0$, x' doesn't change (horizontal movement across $x'' = 0$)

End Conditions -



Restricted - $x(0) = x_0$ and $x(T) = x_T$ (assume $x_T > x_0$); many trajectories are possible (and some you can see aren't possible); each feasible trajectory takes a different amount of time which isn't shown because t isn't on the graph

Optimal Path - there will only be one trajectory that is feasible and takes time T



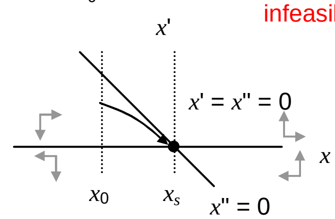
***x*(*T*) Free** - transversality condition $F_{x'}\big|_T = 0 \;\Rightarrow\; f'(x'(T)) = 0$; since f is strictly monotone there is at most one value of x' that satisfies this condition; feasible path goes between x_0 and horizontal line implicitly specified by transversality condition

***T* → ∞ with *x*(*T*) Free** - generates three types of answers:

(a) $x = -\infty$ and $x' = -\infty$; or (b) $x = \infty$ and $x' = \infty$; or

(c) $x = x_s$, $x' = x'' = 0$

We're interested in case (c)... steady-state



Neoclassical Growth Example -

L = number of identical workers

$c(t)$ = consumption per worker at time t

$U(c(t))$ = utility from consumption; $U' > 0$, $U'' < 0$; $\lim_{c \to 0} U'(c) = \infty$ (to avoid corner solutions)

Objective - central planner wants to maximize aggregate discounted utility

$$\max \int_0^\infty e^{-rt} L U(c(t)) dt$$

Production - single output is produced using capital K and labor L

Production Function - $F(K,L)$

Homogeneous of Degree 1 - $F(aK,aL) = aF(K,L)$ (i.e., constant returns to scale)

Both Factors Essential - $F(0,L) = F(K,0) = 0$

Positive, Diminishing Returns - F_K and $F_L > 0$ and F_{KK} and $F_{LL} < 0$

Capital Stock - single output is either consumed or saved to increase stock of capital

$F(K,L) = Lc + K'$ with $K(0) = K_0$ ... this is a <u>constraint</u> for the problem

New Variables - since L grows exponentially, we want to define variables that are per capita (actually per worker): $k \equiv K/L \;\Rightarrow\; f(k) = F(K/L,1)$

From production function properties we know $F(K,L) = Lf(k)$, $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$; to avoid corners solutions we'll add $f'(0) = \infty$ and $f'(\infty) = 0$

Capital Stock Per Worker - use $F(K,L) = Lc + K'$ with $F(K,L) = Lf(k)$ to get

$f(k) = c + k' \;\Rightarrow\; c = f(k) - k'$ so we can rewrite the problem:

$$\max \int_0^\infty e^{-rt} L U(f(k) - k') dt \text{ s.t. } k(0) = k_0,\; k \ge 0,\; f(k) - k' \ge 0$$

Euler Equation -

$F(t, k(t), k'(t)) = e^{-rt} L U(f(k) - k')$

$F_k = e^{-rt} L U'(c) f'(k)$

$F_{k'} = -e^{-rt} L U'(c)$

$$\frac{dF_{k'}}{dt} = re^{-rt}LU'(c) - e^{-rt}LU''(c)c'$$

Euler Equation: $F_x = \frac{dF_{x'}}{dt} \Rightarrow e^{-rt}LU'(c)f'(k) = e^{-rt}L\left[rU'(c) - U''(c)c'\right]$

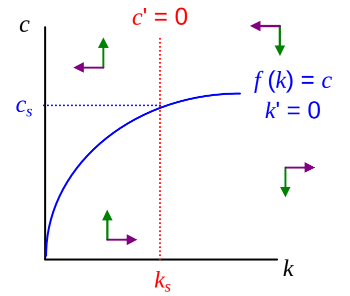
$e^{-rt}L$ cancels $\Rightarrow -\left[f'(k) - r\right] = \frac{U''}{U'}c' = \frac{1}{U'}\frac{dU'(c)}{dt}$ (proportionate rate of change of marginal utility over time)

Solution - has to satisfy two equations:

$c = f(k) - k'$ (balance)

$-\left[f'(k) - r\right] = \frac{U''}{U'}c'$ (Euler)

Develop a phase diagram to study the qualitative properties of the optimal solution



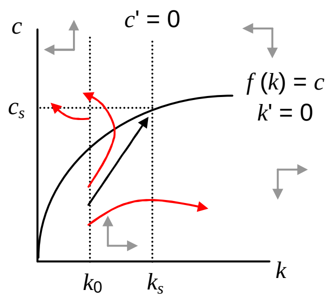
1. **c' = 0** - from Euler equation this means $f'(k) = r$ which has a unique solution k_s
2. **Crossing c' = 0** - $f'(k_s) - r = 0 \Rightarrow f'(k_s + dk) - r < 0$ for $dk > 0$ because $f'(k)$ is monotone decreasing ($f''(k) < 0$) $\therefore k\uparrow \Rightarrow c\downarrow$ (recall that $-U''/U' > 0$)
3. **k' = 0** - from balance equation, this means $f(k) = c$; goes through origin and is increasing and concave because of assumptions about f; allows us to identify $f(k_s) = c_s$
4. **Crossing k' = 0** - $k' = f(k_s) - c_s = 0 \Rightarrow k' = f(k_s) - (c_s + dc) < 0$ for $dc > 0$ $\therefore$ $c\uparrow \Rightarrow k\downarrow$

Steady-State - $k' = c' = 0$

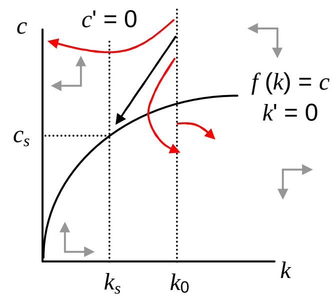
$k_0 < k_s \Rightarrow c' > 0$ and $k' > 0$

$k_0 > k_s \Rightarrow c' < 0$ and $k' < 0$

Result - c & k move monotonically in same direction towards steady-state



Don't reach steady-state



Don't reach steady-state

Neighborhood of Stead-State -

Taylor Series Approximation -

General - $y(x) = y(x_0) + y'(x_0)(x - x_0) + \text{h.o.t.}$ (higher order terms; if we don't include the h.o.t., we replace = with ≈)

Balance Eqn - $k' = f(k) - c$; to make Taylor Series easier to see, let $k' = F(k,c)$

$$k'(t) \approx F(k_s, c_s) + \left.\frac{\partial F}{\partial k}\right|_{k_s}(k - k_s) + \left.\frac{\partial F}{\partial c}\right|_{c_s}(c - c_s)$$

$$k'(t) \approx \left(f(k_s) - c_s\right) + f'(k_s)(k - k_s) - (c - c_s)$$

Since $f(k_s) - c_s = 0$ we have $\boxed{k'(t) \approx f'(k_s)(k - k_s) - (c - c_s)}$

Euler Eqn - $c' = -\frac{U'}{U''}\left[f'(k) - r\right]$; to make Taylor Series easier let $c' = G(k,c)$

$$c'(t) \approx G(k_s, c_s) + \left.\frac{\partial G}{\partial k}\right|_{k_s}(k - k_s) + \left.\frac{\partial G}{\partial c}\right|_{c_s}(c - c_s)$$

$$c'(t) \approx -\frac{U'(c_s)}{U''(c_s)}\left[f'(k_s)-r\right] - \frac{U'(c_s)}{U''(c_s)} f''(k_s)(k-k_s) + (0)(c-c_s)$$

Since $f'(k_s) - r = 0$ we have $\boxed{c'(t) \approx -\frac{U'(c_s)}{U''(c_s)} f''(k_s)(k-k_s)}$

System of DiffEq - $k'(t) \approx f'(k_s)(k-k_s) - (c-c_s)$ and $c'(t) \approx -\frac{U'(c_s)}{U''(c_s)} f''(k_s)(k-k_s)$

Homogeneous Equations - $\bar{k} = k - k_s$ and $\bar{c} = c - c_s$; note that $\bar{k}' = k'$ and $\bar{c}' = c'$ so we can rewrite the original pair of equations... note general form:

$\bar{k}' \approx f'(k_s)\bar{k} - \bar{c} \quad \Rightarrow \quad \bar{k}' = a\bar{k} - \bar{c}$

$\bar{c}' \approx -\frac{U'(c_s)}{U''(c_s)} f''(k_s)\bar{k} \quad \Rightarrow \quad \bar{c}' = b\bar{k}$

General Solution - this type system of differential equations has a general solution (trick 4 form our hints on DiffEq):

$\bar{k} = A_1 e^{m_1 t} + A_2 e^{m_2 t}$

$\bar{c} = B_1 e^{m_1 t} + B_2 e^{m_2 t}$

where m_1 and m_2 are eigenvalues of the coefficient matrix $\begin{bmatrix} a & -1 \\ b & 0 \end{bmatrix}$ (see below)

$B_1 = \frac{(m_1 - a)A_1}{-1}$ and $B_2 = \frac{(m_2 - a)A_2}{-1}$

We get A_1 and A_2 from initial/terminal conditions

Specific Solution - now use those solutions to solve our original diffeq system:

$k = A_1 e^{m_1 t} + A_2 e^{m_2 t} + k_s$

$c = B_1 e^{m_1 t} + B_2 e^{m_2 t} + c_s$

Solving for Constants - find A_1, A_2, B_1 and B_2

Steady-State - $\lim_{t\to\infty} k(t) = k_s$ and $\lim_{t\to\infty} c(t) = c_s$

$k(\infty) = A_1 e^{m_1 \infty} + A_2 e^{m_2 \infty} + k_s = k_s$ ... but $e^{m_1 \infty} = \infty$ so we must have $A_1 = 0$;

$e^{m_2 \infty} = e^{-\infty} = 0$ because $m_2 < 0$... that gives us $k_s = k_s$

Similar reasoning with $c(\infty) = c_s$ gives us $B_1 = 0$

Initial Condition - $k(0) = k_0$ ... $k(0) = A_2 e^{m_2 0} + k_s = A_2 + k_s = k_0 \Rightarrow$

$A_2 = k_0 - k_s$

Use general solution for B_2 above: $B_2 = \frac{(m_2 - a)A_2}{-1} = (a - m_2)(k_0 - k_s)$

Recall that $a = f'(k_s)$ so $B_2 = (f'(k_s) - m_2)(k_0 - k_s)$

Solution - $k(t) = (k_0 - k_s)e^{m_2 t} + k_s$

$c(t) = (f'(k_s) - m_2)(k_0 - k_s)e^{m_2 t} + c_s$

How do k and c change over time near steady-state? take derivative wrt t:

$k'(t) = m_2(k_0 - k_s)e^{m_2 t}$

$c'(t) = m_2(f'(k_s) - m_2)(k_0 - k_s)e^{m_2 t} + c_s$

Note 1 - $m_2 < 0$ and $f'(k_s) > 0$ so $k'(t)$ and $c'(t)$ have same sign (i.e., move in same direction toward steady-state... same result we got with diagram)

Note 2 - we can multiply $k(t)$ and $c(t)$ by m_2 and get:

$$m_2 k(t) = m_2(k_0 - k_s)e^{m_2 t} + m_2 k_s = k'(t) + m_2 k_s \;\Rightarrow\; k'(t) = m_2 k(t) - m_2 k_s$$

$$m_2 c(t) = m_2(f'(k_s) - m_2)(k_0 - k_s)e^{m_2 t} + m_2 c_s \;\Rightarrow\; c'(t) = m_2 c(t) - m_2 c_s$$

$\therefore$ rate of change of k and c are proportional (by m_2) to distance from steady state... these are linear because we used Taylor series approximation so they're only valid near steady-state

What is highest steady-state level of c?

Totally differentiate $c_s = f(k_s)$: $dc_s = f'(k_s)dk_s \;\Rightarrow\; \frac{dc_s}{dk_s} = f'(k_s) > 0$

Totally differentiate $f'(k_s) = r$ (which comes from Euler Equation):

$$f''(k_s)dk_s = dr \;\Rightarrow\; \frac{dk_s}{dr} = \frac{1}{f''(k_s)} < 0$$

$\therefore r\uparrow \Rightarrow k\downarrow \Rightarrow c\downarrow$... so to max c_s we want $r = 0$

Golden Rule - "do unto others as you would have them do unto you"; r is the discount rate of consumption; setting $r = 0$ to maximize cs means we're not discounting future utility of consumption (i.e., consumption of future generations); that means current generation consumes less now (building capital stock for future generations)

Finding Eigenvalues - n x n matrix will have n eigenvalues: m_1, m_2, ... m_n

1. Subtract m from each element in main diagonal
2. Compute determinant with new main diagonal and set it equal to zero
3. All the roots of this equation are the eigenvalues

Example - $\begin{bmatrix} a & -1 \\ b & 0 \end{bmatrix}$

$$\begin{vmatrix} a-m & -1 \\ b & -m \end{vmatrix} = (a-m)m + b = m^2 - am - b = 0$$

Using quadratic formula: $m = \frac{a}{2} \pm \frac{\sqrt{a^2 + 4b}}{2}$

If a, $b > 0$, there will be two real roots... $m_1 > 0$ and $m_2 < 0$

Optimal Control

Why Use It - versus calculus of variations, optimal control
- More generality
- More convenient with constraints (e.g., can put constraints on the derivatives)
- More insights into problem (at least more apparent than through calculus of variations)

General Problem -

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \quad \text{s.t.} \quad x'(t) = g(t, x(t), u(t)),\ t_0,\ t_1,\ x(t_0) = x_0 \text{ fixed; } x(t_1) \text{ free}$$

State Variables - governed by first order differential equations (state equations)

Control Variables - decision variables; assumed to be piecewise continuous (continuous except for finite number of points that aren't); affect objective directly and through state variables

State Equation - differential equation that governs change in a state variable; also called **transition equation**; should have one state equation for each state variable

Functions - f and g are continuously differentiable functions of three independent arguments, none of which is a derivative; in fact, there are no derivatives in optimal control except for the left side of the state equations

Simplest Problem - typical calculus of variation problem looks like:

$$\max_x \int_{t_0}^{t_1} f(t, x(t), x'(t))dt \quad \text{s.t.} \quad x(t_0) = x_0$$

This can be transformed into optimal control problem by using $x'(t) = u$:

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \quad \text{s.t.} \quad x'(t) = u,\ x(t_0) = x_0$$

Transitions - this one was easy, but in general the hardest part is choosing the which variables are state and control variables

Solving -

Hamiltonian - $H(t, x(t), u(t), \lambda(t)) \equiv f(t, x, u) + \lambda g(t, x, u)$ (similar to a lagrangian)

Properties of Optimal Solution - these are necessary conditions

(1) $\frac{\partial H}{\partial u} = f_u + \lambda g_u = 0$	Optimality Condition
(2) $-\frac{\partial H}{\partial x} = \lambda'(t) = -(f_x + \lambda g_x)$	Multiplier Equation
(3) $\frac{\partial H}{\partial \lambda} = x'(t)$	State Equation
(4) $x(t_0) = x_0$	Boundary Condition
(5) $\lambda(t_1) = 0$	Boundary Condition (in this case $x(t_1)$ is free)
(6) $H_{uu} \le 0$ for max (≥ 0 for min)	Second Order Condition

Economic Interpretation - $\lambda(t)$ is marginal valuation of the associated state variable at time t $\therefore$ $\lambda(t_1) = 0$ because $x(t_1)$ is free; in maximization problem, we'd choose $x(t_1)$ iso we can't gain by changing it so marginal benefit of $x(t_1)$ is zero

General Procedure -

[1] Use (1) to find u that maximizes H (same condition for minimization problem)

[2] Solve u from [1] as a function of t, x, and λ: $\bar{u}(t,x,\lambda)$

[3] Substitute u out of (2) and (3) which gives us a system of two differential equations for x and λ

$$\lambda' = -\left[\frac{\partial f(t,x(t),u(t,x,\lambda))}{\partial x} + \lambda\frac{\partial g(t,x(t),u(t,x,\lambda))}{\partial x}\right]$$

$$x'(t) = g(t,x(t),\bar{u}(t,x,\lambda))$$

Equivalent to Euler and Transversality - Proof: look at

$$\max_x \int_{t_0}^{t_1} f(t,x(t),x'(t))dt \quad \text{s.t.} \quad x(t_0) = x_0$$

Already showed equivalent optimal control problem (use $x'(t) = u$)

$H = f(t,x,u) + \lambda u$

From (1): $f_u + \lambda = 0 \;\Rightarrow\; \lambda = -\dfrac{\partial f}{\partial u}$

Differentiate both sides wrt t: $\lambda'(t) = -\dfrac{d}{dt}\dfrac{\partial f}{\partial u}$

From (2): $\lambda'(t) = -(f_x + \lambda g_x) = -\left[\dfrac{\partial f}{\partial x} + \lambda\dfrac{\partial u}{\partial x}\right] = -\dfrac{\partial f}{\partial x}$ (because $\dfrac{\partial u}{\partial x} = 0$)

Combine these to get Euler Equation: $\dfrac{\partial f}{\partial x} = \dfrac{d}{dt}\dfrac{\partial f}{\partial u} = \dfrac{d}{dt}\dfrac{\partial f}{\partial x'}$

From (5): $\lambda(t_1) = -\left.\dfrac{\partial f}{\partial u}\right|_{t_1} = 0$ ... that's the transversality condition: $f_{x'}\big|_{t_1} = 0$

From (6): $H_{uu} \le 0$ ... same as the Legendre condition $f_{x'x'} \le 0$

Example - max $\int_0^5 (ux - u^2 - x^2)dt$ s.t. $x' = x + u$, $x(1) = 2$

$f(t,x,u) = ux - u^2 - x^2$

$g(t,x,u) = x + u$

$H \equiv f(t,x,u) + \lambda g(t,x,u) = ux - u^2 - x^2 + \lambda(x+u)$

Optimal solution properties:

(1) $\dfrac{\partial H}{\partial u} = x - 2u + \lambda = 0$

(2) $-\dfrac{\partial H}{\partial x} = \lambda'(t) = -(f_x + \lambda g_x) = -(u - 2x + \lambda)$

(3) $\dfrac{\partial H}{\partial \lambda} = x'(t) = x + u$

(4) $x(0) = 2$ and (5) $\lambda(5) = 0$

(6) $H_{uu} = -2 \le 0$ (required for max... good)

Solve (1) for u: $u = \dfrac{\mathrm{x} + \lambda}{2}$

Sub into (2) and (3) for system of differential equations:

$$x'(t) = x + \frac{x+\lambda}{2} = \frac{3}{2}x + \frac{1}{2}\lambda$$

$$\lambda'(t) = -\left(\frac{x+\lambda}{2}\right) + 2x - \lambda = \frac{3}{2}x - \frac{3}{2}\lambda$$

From here we use Trick 4 from the DiffEq handout:

General Solution: $x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$

$\lambda(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$ (7) $B_1 = \frac{(r_1 - a_1)A_1}{b_1}$ and (8) $B_2 = \frac{(r_2 - a_1)A_2}{b_2}$

Coefficient Matrix: $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$

Characteristic Equation: $\begin{bmatrix} \frac{3}{2} - r & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} - r \end{bmatrix} = (\frac{3}{2} - r)(-\frac{3}{2} - r) - \frac{3}{4} = 0$ (eigenvalues)

$\frac{9}{4} + r^2 - \frac{3}{4} = r^2 - 3 = 0 \Rightarrow r_1 = \sqrt{3}$ and $r_2 = -\sqrt{3}$

4 Equations & 4 Unknowns - now use (4), (5), (7), (8) to solve for A_1, A_2, B_1, B_2

Several Variables

Each state variable (x_i) needs a state equation: $x_i'(t) = g_i(\bullet)$

Need one multiplier for each state equation: $\lambda_i'(t)$

Each control variable yields one optimality condition: $\frac{\partial H}{\partial u_j} = 0$

control variables could be >, =, or < # of state variables

General Problem - 2 state variables (i = 2) and 2 control variables (j = 2)

$$\max_{u_j} \int_{t_0}^{t_1} f(t, x_1(t), x_2(t), u_1(t), u_2(t))dt$$

s.t. $x_i'(t) = g_i(t, x_1(t), x_2(t), u_1(t), u_2(t))$, $i = 1,2$ (state equations)

$x_1(t_0), x_2(t_0)$ fixed; $x_1(t_1), x_2(t_1)$ free (end conditions)

Hamiltonian - $H(t, x_1(t), x_2(t), u_1(t), u_2(t)) \equiv f + \lambda_1 g_1 + \lambda_2 g_2$

Properties of Optimal Solution -

(1) $\frac{\partial H}{\partial u_j} = \frac{\partial f}{\partial u_j} + \lambda_1 \frac{\partial g_1}{\partial u_j} + \lambda_2 \frac{\partial g_2}{\partial u_j} = 0$, $j = 1,2$

(2) $-\frac{\partial H}{\partial x_i} = \lambda_i'(t) = -\left(\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i}\right)$, $i = 1,2$

(3) $\frac{\partial H}{\partial \lambda_i} = x_i'(t) = g_i(t, x_1(t), x_2(t), u_1(t), u_2(t))$, $i = 1,2$

(4) fixed end conditions from above

(5) $\lambda_i(t_1) = 0$, $i = 1,2$

End Point Conditions

Fixed End Points - modify original problem for both end points fixed:

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \quad \text{s.t.} \quad x'(t) = g(t, x(t), u(t))$$

$$x(t_0) = x_0\,,\ x(t_1) = x_1\,,\ t_0\,,\ t_1 \text{ all fixed}$$

Deriving Necessary Conditions -

Let J^* be max value

$$J^* \equiv \int_{t_0}^{t_1} f(t, x^*, u^*)dt$$

Put $\pm \lambda x^{*}'$, but realize $x^{*}' = g(t, x^*, u^*)$: $J^* = \int_{t_0}^{t_1} [f(t, x^*, u^*) + \lambda g(t, x^*, u^*) - \lambda x^{*}']dt$

What to get ride of the x^{*}' in the last term so solve that with integration by parts:

$$\int_a^b UdV = UV]_a^b - \int_a^b VdU \text{ ... let } U = \lambda \text{ and } dV = x^{*}'dt \quad \Rightarrow \quad dU = \lambda' dt \text{ and } V = x^*$$

$$\int_{t_0}^{t_1} \lambda x^{*}' dt = \lambda x^*\Big|_{t_1} - \lambda x^*\Big|_{t_0} - \int_{t_0}^{t_1} x^* \lambda' dt$$

Plug that back in: $J^* = \int_{t_0}^{t_1} [f(t, x^*, u^*) + \lambda g(t, x^*, u^*) + \lambda' x^*]dt - \lambda x^*\Big|_{t_1} + \lambda x^*\Big|_{t_0}$

Pick a comparison path $J = \int_{t_0}^{t_1} [f(t, x, u) + \lambda g(t, x, u) + \lambda' x]dt - \lambda x\Big|_{t_1} + \lambda x\Big|_{t_0}$

Since J^* is max value: $J - J^* = \Delta J \leq 0$

Sub formulas:

$$\Delta J = \int_{t_0}^{t_1} \{[f(t, x, u) + \lambda g(t, x, u) + \lambda' x] - [f(t, x^*, u^*) + \lambda g(t, x^*, u^*) + \lambda' x^*]\}dt +$$

$$-\lambda x\Big|_{t_1} + \lambda x\Big|_{t_0} + \lambda x^*\Big|_{t_1} - \lambda x^*\Big|_{t_0} \leq 0$$

Since J is a feasible path, it must have the same endpoints as J^* $\therefore$ terms outside integral disappear: $-\lambda x\Big|_{t_1} + \lambda x\Big|_{t_0} + \lambda x^*\Big|_{t_1} - \lambda x^*\Big|_{t_0} = 0$

Define: $L(t, x, u, \lambda) \equiv f(t, x, u) + \lambda g(t, x, u) + \lambda' x$

Sub into ΔJ: $\Delta J = \int_{t_0}^{t_1} [L(t, x, u, \lambda) - L(t, x^*, u^*, \lambda)]dt \leq 0$

This is a difference of the same function so we can use Taylor series to approximate it (note that t and λ are fixed):

$$L(t, x, u, \lambda) - L(t, x^*, u^*, \lambda) = L_x(t, x^*, u^*, \lambda)(x - x^*) + L_u(t, x^*, u^*, \lambda)(u - u^*) + \text{h.o.t.}$$

where $L_x = f_x + \lambda g_x + \lambda'$ and $L_u = f_u + \lambda g_u$

Drop the higher order terms to approximate ΔJ :

$$\delta J = \int_{t_0}^{t_1} \left[(f_x + \lambda g_x + \lambda')(x - x^*) + (f_u + \lambda g_u)(u - u^*) \right] dt \le 0$$

Since this condition holds for arbitrary λ, let $\boxed{\lambda' = -(f_x + \lambda g_x)}$ (multiplier equation)

Now we have $\delta J = \int_{t_0}^{t_1} (f_u + \lambda g_u)(u - u^*) dt \le 0$

Since $u - u^*$ can be >, =, or < 0, we must have $\boxed{f_u + \lambda g_u = 0}$ (optimality condition)

So we end up with the same necessary conditions we had on p.1, except condition (5)

In general condition (5) will be $\left\{ \boxed{\begin{array}{ll} x(t_1) = x_1 & \text{if } x_1 \text{ is fixed} \\ \lambda(t_1) = 0 & \text{if } x_1 \text{ is free} \end{array}} \right.$

Salvage Value - back to original problem and add salvage value to objective function (i.e., value that's only based on ending conditions t_1 and $x(t_1)$); note if both t_1 and $x(t_1)$ are fixed, this salvage value is irrelevant because it would be constant

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t)) dt + \phi(t_1, x(t_1)) \quad \text{s.t.} \quad x'(t) = g(t, x(t), u(t))$$

$$t_0,\ t_1,\ x(t_0) = x_0 \text{ fixed, } x(t_1) = x_1 \text{ free}$$

Deriving Necessary Conditions - 3 step solution

<u>Step 1</u> - arbitrarily choose final state $x(t_1)$ ... we just covered how to solve that in previous section

Let $V^*(x(t_1)) \equiv \int_{t_0}^{t_1} f(t, x^*, u^*) dt$ be max value of integral given $x(t_1)$

Put $\pm \lambda x^{*\prime}$, but realize $x^{*\prime} = g(t, x^*, u^*)$: $V^* = \int_{t_0}^{t_1} \left[f(t, x^*, u^*) + \lambda g(t, x^*, u^*) - \lambda x^{*\prime} \right] dt$

What to get ride of the $x^{*\prime}$ in the last term so solve that with integration by parts just like we did last time so we end up with: (it's on the middle of the previous page)

$$V^* = \int_{t_0}^{t_1} \left[f(t, x^*, u^*) + \lambda g(t, x^*, u^*) + \lambda' x^* \right] dt - \lambda x^* \Big|_{t_1} + \lambda x^* \Big|_{t_0}$$

Let $H(t, x^*, u^*) \equiv f + \lambda g$ so now $V^* = \int_{t_0}^{t_1} \left[H(t, x^*, u^*) + \lambda' x^* \right] dt - \lambda x^* \Big|_{t_1} + \lambda x^* \Big|_{t_0}$

<u>Step 2</u> - show $\lambda(t)$ is marginal value of $x(t)$

Look at t_0 - look at change in V^* with respect to x_0:

$$\frac{\partial V^*}{\partial x_0} = \int_{t_0}^{t_1} \left[H_t \cancel{\frac{dt}{dx_0}} + H_x \frac{dx^*}{dx_0} + H_u \frac{du^*}{dx_0} + H_\lambda \cancel{\frac{d\lambda}{dx_0}} + \lambda' \frac{dx^*}{dx_0} + x^* \cancel{\frac{d\lambda'}{dx_0}} \right] dt + \lambda(t_0)$$

λ was chosen arbitrarily so it's independent of x_0; also time is independent of x_0 $\therefore$ those derivatives are zero

$$\frac{\partial V^*}{\partial x_0} = \int_{t_0}^{t_1}\left[(H_x + +\lambda')\frac{dx^*}{dx_0} + H_u \frac{du^*}{dx_0}\right]dt + \lambda(t_0)$$

We know from necessary conditions that $H_x = f_x + \lambda g_x = -\lambda'$ and $H_u = 0$ $\therefore$ the integral goes away and we have

$\frac{\partial V^*}{\partial x_0} = \lambda(t_0)$ ... which means the change in the maximum value (V^*) caused by a change in the starting condition (x_0) is $\lambda(t_0)$ ... i.e., marginal value of $x(t_0)$

Look at any $t \in (t_0, t_1)$ - just break up the interval; t is initial point for second integral; same as t_0 in previous proof

$$V^*(x(t_1)) \equiv \int_{t_0}^{t_1} f(t, x^*, u^*)dt = \int_{t_0}^{t} f(t, x^*, u^*)dt + \int_{t}^{t_1} f(t, x^*, u^*)dt$$

Look at t_1 - look at change in V^* with respect to x_1:

$$\frac{\partial V^*}{\partial x_1} = \int_{t_0}^{t_1}\left[H_t \cancel{\frac{dt}{dx_1}} + H_x \frac{dx^*}{dx_1} + H_u \frac{du^*}{dx_1} + H_\lambda \cancel{\frac{d\lambda}{dx_1}} + \lambda' \frac{dx^*}{dx_1} + x^* \cancel{\frac{d\lambda'}{dx_1}}\right]dt - \lambda(t_1)$$

λ was chosen arbitrarily so it's independent of x_1; also time is independent of x_1 $\therefore$ those derivatives are zero

$$\frac{\partial V^*}{\partial x_1} = \int_{t_0}^{t_1}\left[(H_x + +\lambda')\frac{dx^*}{dx_1} + H_u \frac{du^*}{dx_1}\right]dt - \lambda(t_1)$$

We know from necessary conditions that $H_x = f_x + \lambda g_x = -\lambda'$ and $H_u = 0$ $\therefore$ the integral goes away and we have

$\frac{\partial V^*}{\partial x_1} = -\lambda(t_1)$ ... which means the change in the maximum value (V^*) caused by a change in the ending condition (x_1) is $-\lambda(t_1)$ ... i.e., marginal value of $x(t_1)$

<u>Step 3</u> - consider $x(t_1)$ being free; if we get to choose $x(t_1)$ obviously we'd want to maximize V^* so we end up with

$\frac{\partial V^*}{\partial x_1} = -\lambda(t_1) = 0$ ... this ignores the salvage value though

To consider the salvage value let $W \equiv \int_{t_0}^{t_1} f(t, x(t), u(t))dt + \phi(t_1, x(t_1))$

Now define $W^*(x(t_1)) = V^* + \phi(t_1, x(t_1))$ is max value for given $x(t_1)$

We want to maximize W^* wrt $x(t_1)$:

$$\frac{\partial W^*}{\partial x(t_1)} = \frac{\partial V^*}{\partial x(t_1)} + \frac{\partial \phi(t_1, x(t_1))}{\partial x(t_1)} = 0 \Rightarrow -\frac{\partial V^*}{\partial x(t_1)} = \boxed{\frac{\partial \phi(t_1, x(t_1))}{\partial x(t_1)} = \lambda(t_1)}$$

which means the marginal benefit of salvage = marginal cost of salvage

Note: if there is no salvage term $\phi = 0$ so this becomes the same as the condition we had before $\lambda(t_1) = 0$

Example - monopoly building product over time and selling at time t_1:

$$\max\ \underbrace{e^{-rt}P(x(t_1))x(t_1)}_{\text{revenue (salvage)}} - \underbrace{\int_{t_0}^{t_1}\left[c_1u^2 + c_2x\right]e^{-rt}dt}_{\text{cost over time}}$$

Endpoint Constraint - back to original problem and add constraint to endpoint

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \quad \text{s.t.} \quad x'(t) = g(t, x(t), u(t)) \text{ and } K(x(t_1)) \geq 0$$

$$t_0,\ t_1,\ x(t_0) = x_0 \text{ fixed, } x(t_1) = x_1 \text{ free}$$

Deriving Necessary Conditions -

Use same methodology as previous derivation to get $V^*(x(t_1))$

Now search for optimal $x(t_1)$ satisfying constraint $K(x(t_1)) \geq 0$

Lagrangian: $L = V^*(x(t_1)) + p(K(x(t_1)))$

KT Conditions:

$$\frac{\partial L}{\partial x(t_1)} = \frac{\partial V^*(x(t_1))}{\partial x(t_1)} + p(K'(x(t_1))) \geq 0 \text{, with equality if } x(t_1) > 0$$

We know $x(t_1) > 0$ and $\frac{\partial V^*}{\partial x(t_1)} = -\lambda(t_1)$ $\quad \therefore \quad \lambda(t_1) = p(K'(x(t_1)))$

Change in constraint at t_1

Marginal value of constraint at t_1

Marginal value of constraint

$$\frac{\partial L}{\partial p} = K(x(t_1)) \geq 0 \text{, with equality if } p > 0 \text{ ... so we get 2 cases:}$$

(a) if not binding ($K(x(t_1)) > 0$), then $p = 0 \ \Rightarrow\ \lambda(t_1) = 0$
(b) if binding, then $K(x(t_1)) = 0$ and $\lambda(t_1) = p(K'(x(t_1)))$

When solving a problem, we usually have to look at both cases and pick the one that works (one probably won't have a solution); if both have a solution, then plug the solution into the objective to see which is better

Free Horizon - back to original problem and let t_1 be free

$$\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))dt \quad \text{s.t.} \quad x'(t) = g(t, x(t), u(t))$$

$$t_0,\ x(t_0) = x_0,\ x(t_1) = x_1 \text{ fixed, } t_1 \text{ free}$$

Deriving Necessary Conditions -

Use same methodology to get $V^*(x_1, t_1) = \int_{t_0}^{t_1}\left[H(t, x^*, u^*) + \lambda' x^*\right]dt - \lambda(t_1)x_1 + \lambda(t_0)x_0$

Now optimize $V^*(x_1, t_1)$ wrt t_1 ... need Leibnitz Rule because t_1 is in integral

$$\frac{\partial V^*}{\partial t_1} = 0 = H(t_1, x^*(t_1), u^*(t_1), \lambda(t_1)) + \lambda'(t_1)x^*(t_1) + \int_{t_0}^{t_1}\left[(H_x + \lambda')\frac{dx^*}{dt_1} + H_u\frac{du^*}{dt_1}\right]dt - \lambda'(t_1)x_1$$

Note terms that cancel

Also at optimum we know $H_x = -\lambda'$ and $H_u = 0$ so integral term goes away

$$H(t_1, x^*(t_1), u^*(t_1), \lambda(t_1)) = \boxed{\left(f + \lambda g\right)\Big|_{t_1} = 0}$$

Optimal Advertising Example -

$a(t) =$ advertising expenditure

$A(t) =$ goodwill (accumulated advertising expenditure

Depreciation of goodwill: $A'(t) = a(t) - \delta A(t)$

Sales: $q(t) = f(p(t), A(t))$

Revenue: $p(t)q(t)$

Revenue net of production cost: $R(p(t), A(t)) = p(t)q(t) - C(q(t))$

Instantaneous profit: $R(p(t), A(t)) - a(t)$

Problem: $\max\limits_{a(t),p(t)} \int_0^\infty e^{-rt}\left[R(p(t), A(t)) - a(t)\right]dt$

s.t. $A'(t) = a(t) - \delta A(t)$, $a(t) \ge 0$, $p(t) \ge 0$, $A(0) = A_0$ fixed

Control Variables - $a(t)$ and $p(t)$

State Variable - $A(t)$

$$H = e^{-rt}(R - a) + \lambda(a - \delta A)$$

(1a) $\dfrac{\partial H}{\partial p} = e^{-rt}\dfrac{\partial R}{\partial p} = 0$

(1b) $\dfrac{\partial H}{\partial a} = -e^{-rt} + \lambda = 0$

(2) $\lambda' = -\dfrac{\partial H}{\partial A} = -e^{-rt}\dfrac{\partial R}{\partial A} + \lambda\delta$

Observations -

(a) price enters integrand (objective), but not state equation $\therefore$ we can use static optimization
Let $\pi(A) \equiv R(p^*(A), A)$
$p^*(A) \equiv \arg\max\limits_p R(p, A)$ (i.e., p^* is value that maximizes R for a given A)

(b) a doesn't enter its own first order condition so we can't solve the traditional wav

Look at problem with only 1 control variable

$$H = e^{-rt}(\pi(A) - a) + \lambda(a - \delta A)$$

(1) $\dfrac{\partial H}{\partial a} = -e^{-rt} + \lambda = 0$

(2) $\lambda' = -\dfrac{\partial H}{\partial A} = -e^{-rt}\dfrac{\partial \pi(A)}{\partial A} + \lambda\delta$

System of Differential Equations:

$A' = a - \delta A$

$\lambda' = -e^{-rt}\dfrac{\partial \pi(A)}{\partial A} + \lambda\delta$ ... can't go further with DiffEq without more details on $\pi(A)$

Can still solve problem though:

(1) $\Rightarrow$ $\lambda = e^{-rt}$ $\Rightarrow$ $\lim\limits_{t\to\infty}\lambda(t) = 0$ and $\lambda' = -re^{-rt}$

Sub $\lambda = e^{-rt}$ into (2): $\lambda' = -e^{-rt}\dfrac{\partial \pi(A)}{\partial A} + \delta e^{-rt}$

Set these equal: $\lambda' = -re^{-rt} = -e^{-rt}\dfrac{\partial \pi(A)}{\partial A} + \delta e^{-rt}$

Cancel e^{-rt}: $-r = -\frac{\partial \pi(A)}{\partial A} + \delta \;\Rightarrow\; \frac{\partial \pi(A)}{\partial A} = r + \delta$

$\therefore$ A^* is constant over time and we want to get to it as soon as possible (jump to it immediately, not gradually)

$A' = 0 = a - \delta A \;\Rightarrow\; a^* = \delta A^*$

Solution: If $A_0 < A^*$ then want to jump to A^* so $a^*(0) = A^* - A_0$

If $A_0 > A^*$ then $a^*(t) = 0$ until $A(t) = A^*$, then $a^*(t) = \delta A$

Summary - $\max_u \int_{t_0}^{t_1} f(t,x,u)dt$

s.t. state equation: $x'(t) = g(t,x,u)$

start time: t_0 fixed

end time: t_1 fixed (unless stated otherwise)

start condition: $x(t_0) = x_0$ fixed

end condition: $x(t_1) = x_1$ free (unless stated otherwise)

$H(t,x,u,\lambda) \equiv f + \lambda g$

Necessary Conditions -

(a) $\frac{\partial H}{\partial \lambda} = x'(t) = g(t,x,u)$ State equation

(b) $\lambda_i' = -\frac{\partial H}{\partial x_i} = -\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x}\right)$ Multiplier (costate, auxiliary, adjoint) equation

(c) $\frac{\partial H}{\partial u} = \frac{\partial f}{\partial u} + \lambda \frac{\partial g}{\partial u} = 0$ Optimality condition

(d) Transversality conditions:

(i) t_1 fixed and $x(t_1) = x_1$ fixed $\Rightarrow$ $x(t_1) = x_1$

(ii) t_1 fixed and $x(t_1)$ free $\Rightarrow$ $\lambda_i(t_1) = 0$

(iii) t_1 free and $x(t_1) = x_1$ fixed $\Rightarrow$ $H\big|_{t_1} = (f + \lambda g)\big|_{t_1} = 0$

(iv) **Endpoint Constraint**: $K(x(t_1)) \geq 0$

2 Cases - solve both; if there is a contradiction in one, it's not a solution; if there is no contradiction, that case is a solution (could have more than one or none)

Case 1 - Binding ($K(x(t_1)) = 0$) $\Rightarrow$ $\lambda(t_1) = pK'(x(t_1))$

Case 2 - Not binding ($K(x(t_1)) > 0$) $\Rightarrow$ $p = 0$ so $\lambda(t_1) = 0$

(v) **Salvage Value**: add $\phi(t_1, x(t_1))$ to objective with t_1 fixed and $x(t_1)$ free

$$\lambda(t_1) = \frac{\partial \phi(t_1, x(t_1))}{\partial x(t_1)}$$

1. From example 7, section I.4 of KS: Let $U(C) = -e^{-\alpha C}$. Since $-U''/U' = \alpha$, $C'(t) = (i - r)/\alpha$.
(a) Assume $K(0) = K_0$, $K(T) = 0$, and $w(t) = 0$, $0 \le t \le T$. Find $C(t)$ and $K(t)$, solving out for any constants using the boundary values.
(b) Discuss the comparative statics of consumption and assets with respect to changes in i and r.

Example 7. An individual seeks the consumption rate at each moment of time that will maximize his discounted utility stream over a period of known length T. The utility of consumption $U(C(t))$ at each moment t is a known increasing concave function (diminishing marginal utility of consumption): $U' > 0$ and $U'' < 0$. Future utility is discounted at rate r. The object is

$$\text{Max} \int_o^T e^{-rt} U(C(t))dt \tag{1.1}$$

subject to a cash flow constraint. The individual derives current income from exogenously determined wages $w(t)$ and from interest earnings iK on his holdings of capital assets $K(t)$. For simplicity, the individual may borrow capital $(K < 0)$ as well as lend it at interest rate i. Capital can be bought or sold at a price of unity. Thus income from interest and wages is allotted to consumption and investment:

$$iK(t) + w(t) = C(t) + K'(t) \tag{1.2}$$

(a) First, note that the consumption and investment equation (1.2) can be rewritten: $C(t) = iK(t) + w(t) - K'(t)$; that means the objective function actually is:

$$\int_0^T e^{-rt} U(iK(t) + w(t) - K'(t))dt \tag{1.3}$$

Now we can use the Euler Equation:

$F(t, K(t), K'(t)) = e^{-rt} U(iK(t) + w(t) - K'(t))$

$F_K(t, K(t), K'(t)) = e^{-rt} U'(C(t))i$ ... e^{-rt} is constant wrt $K(t)$; chain rule on $U(C(t))$ with $\partial C(t)/\partial K(t) = i$

$F_{K'}(t, K(t), K'(t)) = -e^{-rt} U'(C(t))$ ... e^{-rt} is constant wrt $K(t)$; chain rule on $U(C(t))$ with $\partial C(t)/\partial K'(t) = -1$

Euler Equation: $F_x = \dfrac{dF_{x'}}{dt} \Rightarrow e^{-rt} U'(C(t))i = \dfrac{d}{dt}\left(-e^{-rt} U'(C(t))\right)$

Work out the right side: $\dfrac{d}{dt}\left(-e^{-rt} U'(C(t))\right) = re^{-rt} U'(C(t)) - e^{-rt} U''(C(t))C'(t)$

Substitute that back in: $e^{-rt} U'(C(t))i = re^{-rt} U'(C(t)) - e^{-rt} U''(C(t))C'(t)$

Cancel out e^{-rt} and drop the function arguments: $U'i = rU' - U''C'$

Solve for i - r: $i-r=\dfrac{-U''C'}{U'}$ (so far this is all stuff we did in class)

Now look at $U(C)=-e^{-\alpha C}$

$U'(C)=\alpha e^{-\alpha C}$ and $U''(C)=-\alpha^2 e^{-\alpha C}$ so $-U''/U'=\alpha^2 e^{-\alpha C}/\alpha e^{-\alpha C}=\alpha$

Substitute into the Euler equation solution: $i-r=\dfrac{-U''C'}{U'}=\alpha C'$

Which can be rewritten $C'=\dfrac{i-r}{\alpha}$ (this is just explaining stuff that was given)

To solve for $C(t)$ we can integrate both sides of this equation:

$$\int_0^t C'(s)ds=\int_0^t \frac{i-r}{\alpha}ds$$

Working out the integrals gives: $C(t)-C(0)=\left(\dfrac{i-r}{\alpha}\right)t$

Which we can solve for $C(t)$:

$$C(t)=C(0)+\left(\frac{i-r}{\alpha}\right)t \qquad (1.4)$$

Now we have a constant $C(0)$ that we need to solve so we substitute equation (1.4) into the consumption and investment equation (1.2):

$$iK(t)+w(t)=\left[C(0)+\left(\frac{i-r}{\alpha}\right)t\right]+K'(t) \qquad (1.5)$$

Put all the K terms on the left side with $K'(t)$ being positive. Also recall that $w(t)=0$ so that term drops out:

$$K'(t)-iK(t)=-C(0)-\left(\frac{i-r}{\alpha}\right)t \qquad (1.6)$$

This is a differential equation that we need to solve for $K(t)$. It is the same form as the third problem in the differential equation hints handed out in class, where $P=-i$.

Multiply everything in (1.6) by e^{-it}: $e^{-it}K'(t)-ie^{-it}K(t)=-e^{-it}C(0)-\left(\dfrac{i-r}{\alpha}\right)te^{-it}$

We want to integrate both sides to get rid of the $K'(t)$.

$$\int\left[e^{-it}K'(t)-ie^{-it}K(t)\right]dt=\int -e^{-it}C(0)dt-\int\left(\frac{i-r}{\alpha}\right)te^{-it}dt \qquad (1.7)$$

The left side is pretty easy to solve for since $\dfrac{d}{dt}\left(e^{-it}K(t)\right)=e^{-it}K'(t)-ie^{-it}K(t)$

Left Side: $\int\left[e^{-it}K'(t)-ie^{-it}K(t)\right]dt=\int\dfrac{d}{dt}\left(e^{-it}K(t)\right)dt=e^{-it}K(t)$

Right Side: we have to break these up:

$$\int -e^{-it}C(0)dt=\frac{C(0)}{i}e^{-it}$$

$\int -\left(\frac{i-r}{\alpha}\right)te^{-it}dt = -\left(\frac{i-r}{\alpha}\right)\int te^{-it}dt$ ... need to use integration by parts:

$\int Udv = UV - \int VdU$ ... let $U = t$ and $dv = e^{-it}dt$

That means $dU = dt$ and $V = \int e^{-it}dt = \frac{-e^{-it}}{i}$

Plug these into the general formula: $t\frac{-e^{-it}}{i} - \int \frac{-e^{-it}}{i}dt$

Which works out to $t\frac{-e^{-it}}{i} - \frac{e^{-it}}{i^2}$

So the right side of (1.7) works out to: $\frac{C(0)}{i}e^{-it} - \left(\frac{i-r}{\alpha}\right)\left(\frac{-te^{-it}}{i} - \frac{e^{-it}}{i^2}\right)$

Putting those all together, (1.7) works out to:

$$e^{-it}K(t) = \frac{C(0)}{i}e^{-it} - \left(\frac{i-r}{\alpha}\right)\left(\frac{-te^{-it}}{i} - \frac{e^{-it}}{i^2}\right) + \ell$$

where ℓ is an integration constant (actually, it combines three different integration constants).

We can cancel out the e^{-it}: $K(t) = \frac{C(0)}{i} - \left(\frac{i-r}{\alpha}\right)\left(\frac{-t}{i} - \frac{1}{i^2}\right) + \ell e^{it}$

Now multiply out the second term:

$$K(t) = \frac{C(0)}{i} + \frac{ti^2 + i - tir - r}{\alpha i^2} + \ell e^{it} \qquad (1.8)$$

Equations 1.4 and 1.8 give us two equations and two unknowns ($C(0)$ and ℓ). We now use the starting and ending capital values to solve these equations:

$K(0) = K_0 \;\Rightarrow\; K(0) = \frac{C(0)}{i} + \frac{(0)i^2 + i - (0)ir - r}{\alpha i^2} + \ell e^{i(0)} = K_0$

Remove the terms that are zero or one: $\frac{C(0)}{i} + \frac{i-r}{\alpha i^2} + \ell = K_0$

Solve for ℓ:

$$\ell = K_0 - \frac{C(0)}{i} - \frac{i-r}{\alpha i^2} \qquad (1.9)$$

$K(T) = 0 \;\Rightarrow\; K(T) = \frac{C(0)}{i} + \frac{Ti^2 + i - Tir - r}{\alpha i^2} + \ell e^{iT} = 0$

Substitute (1.9): $\frac{C(0)}{i} + \frac{Ti^2 + i - Tir - r}{\alpha i^2} + \left[K_0 - \frac{C(0)}{i} - \frac{i-r}{\alpha i^2}\right]e^{iT} = 0$

Multiply out e^{iT}: $\frac{C(0)}{i} + \frac{Ti^2 + i - Tir - r}{\alpha i^2} + K_0 e^{iT} - \frac{C(0)}{i}e^{iT} - \left(\frac{i-r}{\alpha i^2}\right)e^{iT} = 0$

Put $C(0)$ terms on left side and others on right:

$$\frac{C(0)}{i} - \frac{C(0)}{i}e^{iT} = \left(\frac{i-r}{\alpha i^2}\right)e^{iT} - \frac{Ti^2 + i - Tir - r}{\alpha i^2} - K_0 e^{iT}$$

Solve for $C(0)$: $C(0)\left(\frac{1-e^{iT}}{i}\right) = \left(\frac{i-r}{\alpha i^2}\right)e^{iT} - \frac{Ti^2 + i - Tir - r}{\alpha i^2} - K_0 e^{iT}$

Which "simplifies" to:

$$C(0) = \left[\left(\frac{i-r}{\alpha i^2}\right)e^{iT} - \frac{Ti^2 + i - Tir - r}{\alpha i^2} - K_0 e^{iT}\right]\left[\frac{i}{1-e^{iT}}\right] \quad (1.10)$$

The "solution" therefore, incorporates (1.4), (1.8), (1.9), (1.10)

$$C(t) = C(0) + \left(\frac{i-r}{\alpha}\right)t$$

$$K(t) = \frac{C(0)}{i} + \frac{ti^2 + i - tir - r}{\alpha i^2} + \ell e^{it}$$

$$\ell = K_0 - \frac{C(0)}{i} - \frac{i-r}{\alpha i^2}$$

$$C(0) = \left[\left(\frac{i-r}{\alpha i^2}\right)e^{iT} - \frac{Ti^2 + i - Tir - r}{\alpha i^2} - K_0 e^{iT}\right]\left[\frac{i}{1-e^{iT}}\right]$$

(b) Comparative statics of consumption (C) and assets (K) with respect to changes in i an r...

$\frac{\partial C(t)}{\partial i} = \frac{\partial C(0)}{\partial i} + \frac{t}{\alpha}$ ... solving $\frac{\partial C(0)}{\partial i}$ is left to the interested reader

$\frac{\partial C(t)}{\partial r} = \frac{\partial C(0)}{\partial r} - \frac{t}{\alpha}$ ... solving $\frac{\partial C(0)}{\partial r}$ is left to the interested reader

The math on these are pretty ridiculous (or I have some errors that I couldn't track down). Realistically, we would expect

$$\frac{\partial C(t)}{\partial i} < 0 \ , \ \frac{\partial C(t)}{\partial r} > 0 \ , \ \frac{\partial K(t)}{\partial i} > 0 \ , \ \frac{\partial K(t)}{\partial r} < 0$$

That is, if interest rates rise, the individual would consume less now (i.e., save more) in the hopes of higher returns from his assets and increased consumption in the future (so $i\uparrow \Rightarrow C\downarrow$ and $K\uparrow$). If the discount rate rises, future consumption brings less utility so the individual would consume more now and save less (so $r\downarrow \Rightarrow C\uparrow$ and $K\downarrow$).

2. KS I.4, exercise 2, p.28. Find the Euler equation and its solution for

$$\int_a^b F(t,x,x')dt \quad \text{subject to} \quad x(a)=A\,,\ x(b)=B$$

but do not evaluate constants of integration where

(a) $F(t,x,x') = (x')^2 / t^3$

(b) $F(t,x,x') = (x')^2 - 8xt + t$

(a) $F_x = 0$

$F_{x'} = 2x'/t^3$

Euler equation: $F_x = \frac{d}{dt}F_{x'} \ldots\ 0 = \frac{d}{dt}\left(2x't^3\right)$

Integrate both sides: $0 = \int \frac{d}{dt}\left(2x't^{-3}\right)dt = 2x't^3 + c_1$

Solve for x': $x' = \frac{-c_1t^{-3}}{2}$ (c_1 is an integration constant)

Integrate both sides to solve for x: $\int x'dt = \int \frac{-ct^{-3}}{2}dt$

Which works out to: $\boxed{x(t) = \frac{c_1t^{-4}}{8} + c_2}$ (c_1 and c_2 are integration constants)

(b) $F_x = -8t$

$F_{x'} = 2x'$

Euler equation: $F_x = \frac{d}{dt}F_{x'} \ldots\ -8t = \frac{d}{dt}(2x')$

Integrate both sides: $\int -8tdt = \int \frac{d}{dt}(2x')dt$

Which works out to: $-4t^2 + c_1 = 2x'$

Integrate both sides to solve for x: $\int 2x'dt = \int \left(c_1 - 4t^2\right)dt$

Which works out to: $\boxed{x(t) = \frac{c_1t}{2} - \frac{2}{3}t^3 + c_2}$ (c_1 and c_2 are integration constants)

3. KS I.4, exercise 4, p.28. Solve example 6 for the case r = 0. Explain the economic reason for your answer.

Example 6. Continuing, suppose production cost is a monotone increasing, convex function $g(x')$ of the production rate x':

$$g(0)=0\,,\quad g'\geq 0\,,\quad g''>0 \quad \text{for } x\geq 0$$

The quadratic production cost is clearly a special case. For the problem

$$\text{Min } \int_0^T e^{-rt}\left[g(x')+c_2 x\right]dt \quad \text{subject to} \quad x(0)=0\,,\quad x(T)=B$$

Compute

$$F_x = e^{-rt}c_2\,,\quad F_{x'}=e^{-rt}g'(x')\,,\quad dF_{x'}/dt=-re^{-rt}g'(x')+e^{-rt}g''(x')x''$$

The Euler equation is therefore

$$\left.\begin{aligned} F_x &= dF_{x'}/dt \\ e^{-rt}c_2 &= -re^{-rt}g'(x')+e^{-rt}g''(x')x'' \\ c_2 &= -rg'(x')+g''(x')x'' \end{aligned}\right\}\text{Steps left out of textbook}$$

$$g''(x'(t))x''(t)=rg'(x'(t))+c_2$$

x' is the production rate

$g(x')$ is the production cost

$\dfrac{\partial g(x')}{\partial x'}=g'(x')$ is the marginal cost of production

$\dfrac{\partial g'(x')}{\partial t}=g''(x')x''$ is the time rate of change of marginal cost of production

When r = 0, $g''(x'(t))x''(t)=c_2$, so the time rate of change of MC is constant and equal to the "unit cost of holding inventory per unit time" (explained in Example 1 on p.5). That means marginal cost of production when plotted over time is linear with slope equal to the marginal cost of inventory.

Another way to look at the problem is to consider that r is the discount rate for expenditures. When r = 0, it doesn't matter when we produce (or store the output). All we have to do is balance out the production costs and the holding costs without the complicated time value of money stuff.

Documentation

Chifeng Dai went over all the problems with me in his office (very patiently over about an hour and a half). I pretty much went all through problems 2 and 3 with him watching my work. I worked out the basic strategy for problem 1, but I had a couple of errors that made it look much simpler in his office. When I fixed the errors, the problem was much more complicated and I couldn't solve it.

I reviewed my work with Jon Parker. We had loads of fun with number 1 and even after we "finished" couldn't figure out if we did it right.

1. KS, p.56, exercise 8.3. Find extremals for

$$\int_0^1 \left\{ \frac{1}{2}[x'(t)]^2 + x(t)x'(t) + x(t) \right\} dt$$

when $x(0)$ and $x(1)$ may be chosen freely.

There's a slight difference in this problem that we didn't cover in class: both endpoints are free. (Section I.8 only talked about the end condition, not the initial condition.) So just for grins, let's look at what the transversality condition would be if both end points are free to take any value. We'll start from equation (3) on p.53:

$$g'(0) = \int_{t_0}^{t_1} [F_x(t,x,x')h + F_{x'}(t,x,x')h']dt = 0$$

As we did in class, we'll break up the integral:

$$g'(0) = \int_{t_0}^{t_1} F_x(t,x,x')hdt + \int_{t_0}^{t_1} F_{x'}(t,x,x')h'dt = 0$$

Then use integration by parts to solve the term on the right (to get rid of the h'):

Here's the formula for integration by parts: $\int UdV = UV - \int VdU$

In this case, we'll use:

$$U = F_{x'} \ \Rightarrow \ dU = dF_{x'} \quad \text{and} \quad dV = h'dt = \frac{dh}{dt}dt = dh \ \Rightarrow \ V = h$$

So based on the formula:

$$\int_{t_0}^{t_1} F_{x'}(t,x,x')h'dt = F_{x'}h\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} h\frac{dF_{x'}}{dt}dt = F_{x'}h\Big|_{t_1} - F_{x'}h\Big|_{t_0} - \int_{t_0}^{t_1} h\frac{dF_{x'}}{dt}dt$$

That last step was tricky because we added dt/dt. In class we were able to remove the middle term because $h(t_0) = 0$, but that's not the case now.

So, plugging that back into $g'(0)$, we get:

$$g'(0) = \int_{t_0}^{t_1} F_x hdt + F_{x'}h\Big|_{t_1} - F_{x'}h\Big|_{t_0} - \int_{t_0}^{t_1} h\frac{dF_{x'}}{dt}dt = 0$$

Now we can combine the integrals:

$$g'(0) = F_{x'}h\Big|_{t_1} - F_{x'}h\Big|_{t_0} + \int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)hdt = 0$$

Since we originally picked any admissible function $x(t)$, we could've picked one with the same endpoints as $x^*(t)$ so we could have $h(t_0) = 0$ and $h(t_1) = 0$. That means we must have:

$$\int_{t_0}^{t_1}\left(F_x - \frac{dF_{x'}}{dt}\right)h(t)dt = 0$$

This leads us to the Euler equation:

$$F_x = \frac{dF_{x'}}{dt} \quad \text{for } t_0 \le t \le t_1$$

Since the Euler equation holds for all $t \in [t_0,t_1]$, we can drop that term and conclude that:

$$F_{x'}h\Big|_{t_1} - F_{x'}h\Big|_{t_0} = 0$$

Again, we use the trick of deciding which admissible function we selected. Suppose $x(t)$ had the same initial condition as $x^*(t)$ (i.e., $h(t_0) = 0$), but not the same end condition. That means we must have:

$$F_{x'}h\Big|_{t_1} = 0 \;\Rightarrow\; F_{x'}\Big|_{t_1} = 0$$

Here's the big new step. We can use a similar argument for $x(t)$ and $x^*(t)$ having the same end condition with a different initial condition (i.e., $h(t_1) = 0$). That leads us to:

$$F_{x'}h\Big|_{t_0} = 0 \;\Rightarrow\; F_{x'}\Big|_{t_0} = 0$$

So the **transversality conditions** for a problem with freely chosen initial and ending conditions are:

$$\boxed{F_{x'}\Big|_{t_1} = 0 \text{ and } F_{x'}\Big|_{t_0} = 0}$$

Now we can get back to solving the problem. Given the integral we can determine:

$$F(t,x,x') = \frac{1}{2}\left[x'(t)\right]^2 + x(t)x'(t) + x(t)$$
$$F_x = x'(t) + 1$$
$$F_{x'} = x'(t) + x(t)$$

Now we apply the Euler equation:

$$F_x = \frac{d}{dt}F_{x'} \;\Rightarrow\; x'(t) + 1 = \frac{d}{dt}\left(x'(t) + x(t)\right)$$

Rather than work out the derivative, we'll just take the integral of both sides:

$$\int\{x'(t)+1\}dt = \int\left\{\frac{d}{dt}\left(x'(t)+x(t)\right)\right\}dt$$

These are actually easy to work out. We'll combine the integration constants into a single value (c_1) and put it on the right side:

$$x(t) + t = x'(t) + x(t) + c_1$$

Now rearrange terms to get $x'(t)$ on the left and all other terms on the right:

$$x'(t) = t - c_1$$

This is a pretty simple first order differential equation... the kind we hope to get on the exam (hint). All we have to do is integrate both sides:

$$\int x'(t)dt = \int (t - c_1)dt$$

Again, we'll combine the integration constants into a single value (c_2) and put it on the right side:

$$x(t) = \frac{t^2}{2} - c_1 t + c_2$$

Now we use the transversality conditions to develop a system of equations to solve for the integration constants. Recall that $t_1 = 1$ and $t_0 = 0$.

$$F_{x'}\big|_{t_1} = 0 \;\Rightarrow\; F_{x'}\big|_{t_1} = x'(1) + x(1) = (1 - c_1) + \left(\frac{1^2}{2} - c_1 \cdot 1 + c_2\right) = 0$$

$$F_{x'}\big|_{t_0} = 0 \;\Rightarrow\; F_{x'}\big|_{t_0} = x'(0) + x(0) = (0 - c_1) + \left(\frac{0^2}{2} - c_1 \cdot 0 + c_2\right) = 0$$

These equations simplify down to:

$$2c_1 - c_2 = \frac{3}{2}$$

$$c_1 - c_2 = 0$$

Which are pretty easy to solve:

$$c_1 = c_2 \;\Rightarrow\; 2c_1 - c_1 = \frac{3}{2} \;\Rightarrow\; c_1 = \frac{3}{2}$$

Therefore, the extremal for the problem is:

$$\boxed{x(t) = \frac{t^2}{2} - \frac{3t}{2} + \frac{3}{2}}$$

2. KS, p.63, exercise 9.1.d. Let B be the total quantity of some exhaustible resource, for example, the amount of mineral in a mine, controlled by a monopolist who discounts continuously at rate r and wishes to maximize the present value of profits from the mine. Let $x(t)$ be the cumulative amount sold by time t and $x'(t)$ be the current rate of sales. The net price (gross price less cost of mining) $p(x')$ is assumed to be a decreasing, continuously differentiable function of the current rate of sales:

$$p'(x') < 0$$

(a) Let T denote the time at which the resource will be depleted. Then choose $x(t)$ and T to maximize

$$\int_0^T e^{-rt} p(x'(t))x'(t)dt \text{ subject to } x(0) = 0 \text{ and } x(T) = B$$

(d) Now suppose that the net price depends on the cumulative amount extracted and sold, as well as on the current rate of production--sales. (Mining may become more expensive as the digging goes deeper. Alternatively, the resource may be durable--e.g., aluminum--and the amount previously sold may be available for a second-hand market, affecting the demand for newly mined resource.) In particular, suppose

$$p(x', x) = a - bx - cx'$$

where a, b, and c are given positive constants. Find the sales plan to maximize the present value of the profit stream in this case. The constants of integration and T may be stated implicitly as the solution of a simultaneous system of equations.

With the new price function, the problem becomes:

$$\int_0^T e^{-rt} p(x', x)x'\,dt = \int_0^T e^{-rt}(a - bx - cx')x'\,dt = \int_0^T e^{-rt}\left(ax' - bxx' - cx'^2\right)dt$$

Given the integral we can determine:

$$F(t, x, x') = e^{-rt}\left(ax' - bxx' - cx'^2\right)$$
$$F_x = -e^{-rt}bx'$$
$$F_{x'} = e^{-rt}(a - bx - 2cx')$$

Now we apply the Euler equation:

$$F_x = \frac{d}{dt}F_{x'} \;\Rightarrow\; -e^{-rt}bx' = \frac{d}{dt}\left[e^{-rt}(a - bx - 2cx')\right]$$

Normally, we'd just integrate both sides, but the term on the left is a little too tricky for my little brain. Instead, we'll work out the derivative on the right so we can cancel out the e^{-rt}:

$$-e^{-rt}bx' = -re^{-rt}(a - bx - 2cx') + e^{-rt}(-bx' - 2cx'') = e^{-rt}\left[-ra + rbx + 2rcx' - bx' - 2cx''\right]$$

Now we can cancel out the e^{-rt} and move all the terms with x to the left side:

$$2cx'' - 2rcx' - rbx = -ra$$

This is an ugly second order differential equation. The outline for solving it is on pages 332-334 of the text.

Complete equation: $2cx''-2rcx'-rbx=-ra$
Reduced equation: $2cx''-2rcx'-rbx=0$
Characteristic Equation: $ce^{\lambda t}(\lambda^2+A\lambda+B)=0$, where $A=-2rc$, $B=-rb$
We must have $\lambda^2+A\lambda+B=\lambda^2-2rc\lambda-rb=0$
Since r, b, and c are positive, we know $A^2>4B$ (i.e., $4r^2c^2>-rb$). That means the general solution to the reduced equation is

$$x(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$$

where $\lambda_1,\lambda_2=\dfrac{-A\pm\sqrt{A^2-4B}}{2}=\dfrac{2rc\pm\sqrt{4r^2c^2+4rb}}{2}=rc\pm\sqrt{r^2c^2+rb}$
Now to find the solution to the complete equation, we need to find a specific solution to it. One method to do this is to let $x(t)$ be a constant. In this case $x(t)=a/b$. To verify that this works, realize that $x'=x''=0$ and $-rb(a/b)=-ra$. Based on Theorem 1 on page 332 of KS, "the general solution of the complete equation is the sum of any particular solution of the complete equation and the general solution of the reduced equation." Therefore, our solution is:

$$\boxed{\begin{array}{c} x(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}+\dfrac{a}{b} \\ \text{where } \lambda_1,\lambda_2=rc\pm\sqrt{r^2c^2+rb}\end{array}}$$

To find the system of equations for the constants $c1$ and $c2$, we turn to the transversality conditions. In this case, both initial and ending conditions are set ($x(0)=0$ and $x(T)=B$) and the initial horizon is set ($t_0=0$). The only thing that is free to vary is the ending horizon ($t_1=T$, a decision variable). That means we must have:

$$F-x'F_{x'}\big|_T=0$$

This part can get hard to follow, but we're finally down to "simple" algebra and plug and chug math. Plug in the formulas from page 4 for F and Fx' and indicate that x and x' are evaluated at T:

$$F-x'F_{x'}\big|_T=\left\{e^{-rT}\left(ax'(T)-bx(T)x'(T)-c[x'(T)]^2\right)\right\}-\left\{e^{-rT}\left(a-bx(T)-2cx'(T)\right)x'(T)\right\}=0$$

Now we can factor out the e^{-rT} and substitute $x(T)=B$ and multiply out the second term:

$$ax'(T)-bBx'(T)-c[x'(T)]^2-a+bB+2c[x'(T)]^2=0$$

It's amazing how this all cancels out so that intermediate step was included to keep this from being too confusing. Everything boils down to this:

$$c[x'(T)]^2=0 \quad\Rightarrow\quad x'(T)=0$$

Yes, after all that work, it seems this would be an obvious result since we know $x(T)=B$. Since B is a constant, naturally, $x'(T)=0$.

Now for that system of equations. We have three unknowns (c_1, c_2, and T) and three conditions:

$$x(0) = 0 \;\Rightarrow\; x(0) = c_1 + c_2 + \frac{a}{b} = 0$$

$$x(T) = B \;\Rightarrow\; x(T) = c_1 e^{\lambda_1 T} + c_2 e^{\lambda_2 T} + \frac{a}{b} = B,$$

$$x'(T) = 0 \;\Rightarrow\; x'(T) = c_1 \lambda_1 e^{\lambda_1 T} + c_2 \lambda_2 e^{\lambda_2 T} = 0$$

where $\lambda_1, \lambda_2 = rc \pm \sqrt{r^2 c^2 + rb}$

Given actual values for a, b, c, and r, it should be fairly easy (relative to all this other math) to solve for c_1 and c_2. Fortunately, the problem doesn't make us solve for these unknowns because it would be pretty hard solve for T.

Documentation.

I reviewed both problems with Prof. Dai to make sure I knew what I was doing. He reviewed how to derived the transversality condition in problem 1 for the free initial condition. For the second problem, he confirmed that it would be easier to differentiate $dF_{x'}/dt$ and then went over solving the differential equation with me and pointed me to the appendix in the book that covered it. He also reminded me to clearly show the third equation since the unknowns are c_1, c_2, and T.

1. KS II.2, exercise 5, p.131. Solve by optimal control

$$\min \int_0^1 u^2(t)dt$$

subject to $x'(t) = x(t) + u(t)$, $x(0) = 1$

$f(t,x,u) = u^2$
$g(t,x,u) = x + u$
$H \equiv f(t,x,u) + \lambda g(t,x,u) = u^2 + \lambda(x+u)$

Optimal solution properties:

(1) $\frac{\partial H}{\partial u} = 2u + \lambda = 0$ (4) $x(0) = 1$

(2) $-\frac{\partial H}{\partial x} = \lambda'(t) = -(f_x + \lambda g_x) = -\lambda$ (5) $\lambda(1) = 0$

(3) $\frac{\partial H}{\partial \lambda} = x'(t) = x + u$ (6) $H_{uu} = 2 \geq 0$ (required for min... good)

Solve (1) for u: $u = -\frac{1}{2}\lambda$

Sub into (2) and (3) for system of differential equations:

$x'(t) = x - \frac{1}{2}\lambda$
$\lambda'(t) = -\lambda$

General Solution: $x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$
$\lambda(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$ $B_1 = \frac{(r_1 - a_1)A_1}{b_1}$ and $B_2 = \frac{(r_2 - a_1)A_2}{b_2}$

Coefficient Matrix: $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -1 \end{bmatrix}$

Eigenvalues: $\begin{bmatrix} 1-r & -\frac{1}{2} \\ 0 & -1-r \end{bmatrix} = (1-r)(-1-r) = 0 \Rightarrow r_1 = 1$ and $r_2 = -1$

Solve for B_1: $B_1 = \frac{(1-1)A_1}{b_1} = 0$

Use (5) to solve for B_2: $\lambda(1) = B_1 e^{r_1(1)} + B_2 e^{r_2(1)} = 0 + B_2 e^{-1} = 0 \Rightarrow B_2 = 0$

Solve for A_2: $B_2 = 0 = \frac{(r_2 - a_1)A_2}{b_2} = \frac{(-1-1)A_2}{-1} \Rightarrow A_2 = 0$

Use (4) to solve for A_1: $x(0) = A_1 e^{r_1(0)} + A_2 e^{r_2(0)} = A_1 = 1$

Solution: $\boxed{\begin{array}{l} x(t) = e^t \\ \lambda(t) = 0 \\ u(t) = 0 \end{array}}$

2. KS II.2, exercise 9, p.132. The Euler equation for the calculus of variations problem

$$\max \int_{t_0}^{t_1} F(x, x')dt$$

subject to $x(t_0) = x_0$ fixed, $x(t_1) = x_1$ free

can be written

$$F - x' F_{x'} = \text{const}, \quad t_0 \le t \le t_1$$

Show that this is equivalent to the condition that the Hamiltonian for the related control problem is constant.

Control problem: max $\int_{t_0}^{t_1} F(x,u)dt$ s.t. $x'(t) = u$ $x(t_0) = x_0$ fixed, $x(t_1) = x_1$ free

Hamiltonian: $H \equiv f(t,x,u) + \lambda g(t,x,u) = F(x,u) + \lambda u$

Property (1): $\dfrac{\partial H}{\partial u} = \dfrac{\partial F}{\partial u} + \lambda = 0 \quad \Rightarrow \quad \dfrac{\partial F}{\partial u} = -\lambda$

From the constraint $x'(t) = u$ we can say:

$\dfrac{\partial F}{\partial u} = \dfrac{\partial F}{\partial x'} = F_{x'} = -\lambda$ (last step substitutes form property (1) above)

$F(x,u) = F(x,x')$

Substitute these into H: $H \equiv F(x,u) + \lambda u = F(x,x') - F_{x'}x' = \text{constant}$

3. Consider the dynamical system:

$$x' = -4x + 3y$$
$$y' = 3x - 5y$$

a) What is the steady state of the system?
b) Find the roots of the characteristic equation.
c) What is the general solution to this system of equations? Is this system stable? (Hint a system is stable if $\lim_{t\to\infty} x(t) = x_s$ and $\lim_{t\to\infty} y(t) = y_s$.)
d) Draw the phase diagram for this system. Verify your answer to (c) about the type of stability of this dynamical system.

a) Steady state means $x' = y' = 0$ (KS p.346)

(i) $x' = -4x + 3y = 0$

(ii) $y' = 3x - 5y = 0$

Solve (i) for y: $y = \frac{4}{3}x$

Sub into (ii): $3x - 5\left(\frac{4}{3}x\right) = 0 \quad \Rightarrow \quad x = 0$

Solve for y: $y = \frac{4}{3}x = 0 \quad \Rightarrow$ steady state: $\boxed{x = 0 \text{ and } y = 0}$

b) Characteristic equation: $x'' - (a_1 + b_2)x' + (a_1 b_2 - b_1 a_2)x = 0$ (KS p.344)

Roots: $r^2-(a_1+b_2)r+a_1b_2-b_1a_2=0$

Coefficient Matrix: $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}=\begin{bmatrix} -4 & 3 \\ 3 & -5 \end{bmatrix}$

Sub values for roots: $r^2-(-4-5)r+(-4)(-5)-(3)(3)=r^2+9r+11=0$

Or use eigenvalues: $\begin{bmatrix} -4-r & 3 \\ 3 & -5-r \end{bmatrix}=(-4-r)(-5-r)-9=$

$(20+4r+5r+r^2)-9=r^2+9r+11=0$

Solve for r: $r=\frac{-9}{2}\pm\frac{\sqrt{81-4(11)}}{2}=\frac{-9\pm\sqrt{37}}{2}\Rightarrow \boxed{r_1=-5.959 \text{ and } r_2=-12.041}$

c) General Solution: $\boxed{\begin{aligned} x(t)&=A_1e^{r_1t}+A_2e^{r_2t} \\ y(t)&=B_1e^{r_1t}+B_2e^{r_2t}\end{aligned}}$ $B_1=\frac{(r_1-a_1)A_1}{b_1}$ and $B_2=\frac{(r_2-a_1)A_2}{b_2}$

A_1 and A_2 determined by initial and ending conditions (KS p.346); since we're not given these conditions we cannot solve further.

$\lim_{t\to\infty} x(t)=A_1e^{-\infty}+A_2e^{-\infty}=0$ constant

$\lim_{t\to\infty} y(t)=B_1e^{-\infty}+B_2e^{-\infty}=0$ constant $\therefore$ $\boxed{\text{system is stable}}$

Another check: <u>Routh-Hurwitz</u>: If $r_2<r_1<0$, then $a_1b_2-a_2b_1>0$ and $a_1+b_2<0$ are sufficient for stability (KS p.347).

$r_2<r_1<0$: $-12.041<-5.959<0$ ... yes

$a_1b_2-a_2b_1>0$: $(-4)(-5)-(3)(3)=20-9=11>0$ ... yes

$a_1+b_2<0$: $-4-5=-9<0$ ... yes $\therefore$ system is stable

d) Plot on (x,y) plane

$x'=-4x+3y=0 \Rightarrow$ $\boldsymbol{y=\,^4/_3\,x}$

Must cross this line vertically ($x'=0$)

$x'=-4x+3(y+dy)>0$ for $dy>0$ $\therefore$

$x\uparrow$ above $y=\,^4/_3\,x$

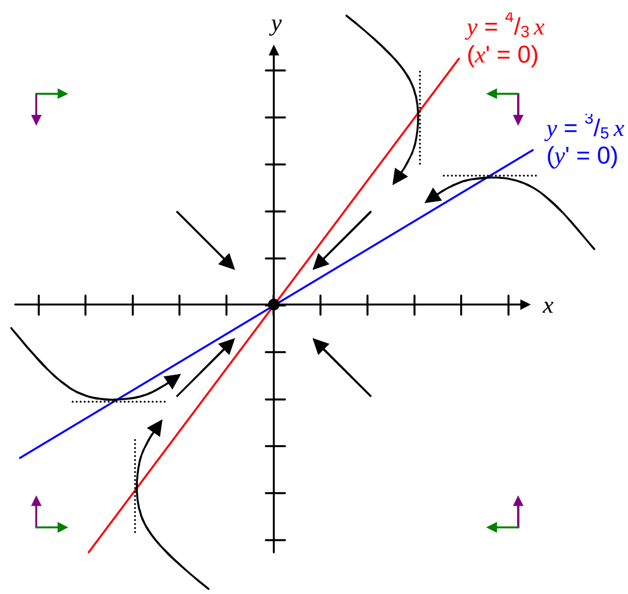
$y'=3x-5y=0 \Rightarrow$ $\boldsymbol{y=\,^3/_5\,x}$

Must cross this line horizontally ($y'=0$)

$y'=3(x+dx)-5y>0$ for $dx>0$ $\therefore$

$y\uparrow$ right of $y=\,^3/_5\,x$

Note on graph, once path enters narrow quadrants, they can't get out (because have to cross blue line horizontally and red line vertically), $\therefore$ they are funneled to the origin. System is stable.



Documentation.

I reviewed my work with Prof Dai. He pointed out that $F(x,x')=F(x,u)$ in #2 (I didn't realize it was the same function).

1. KS II.6.1, p.154.

minimize $\int_0^1 u^2(t)dt$

subject to $x'(t) = x(t) + u(t)$, $x(0) = 1$, $x(1) = 0$

$f(t,x,u) = u^2$

$g(t,x,u) = x + u$

$H \equiv f(t,x,u) + \lambda g(t,x,u) = u^2 + \lambda(x+u)$

Optimal solution properties:

(1) $\frac{\partial H}{\partial u} = 2u + \lambda = 0$ (4) $x(0) = 1$

(2) $-\frac{\partial H}{\partial x} = \lambda'(t) = -(f_x + \lambda g_x) = -\lambda$ (5) $x(1) = 0$

(3) $\frac{\partial H}{\partial \lambda} = x'(t) = x + u$ (6) $H_{uu} = 2 \geq 0$ (required for min... good)

Solve (1) for u: $u = -\frac{1}{2}\lambda$

Sub into (2) and (3) for system of differential equations:

$x'(t) = x - \frac{1}{2}\lambda$

$\lambda'(t) = -\lambda$

General Solution: $x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$

$\lambda(t) = B_1 e^{r_1 t} + B_2 e^{r_2 t}$ $B_1 = \frac{(r_1 - a_1)A_1}{b_1}$ and $B_2 = \frac{(r_2 - a_1)A_2}{b_2}$

Coefficient Matrix: $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -1 \end{bmatrix}$

Eigenvalues: $\begin{bmatrix} 1-r & -\frac{1}{2} \\ 0 & -1-r \end{bmatrix} = (1-r)(-1-r) = 0 \Rightarrow r_1 = 1$ and $r_2 = -1$

Solve for B_1: $B_1 = \frac{(1-1)A_1}{b_1} = 0$

Use (4): $x(0) = A_1 e^0 + A_2 e^0 = A_1 + A_2 = 1 \Rightarrow A_2 = 1 - A_1$

Use (5) to solve for A_2: $x(1) = A_1 e^1 + A_2 e^{-1} = (1 - A_2)e^1 + A_2 e^{-1} = 0 \Rightarrow A_2 = \frac{e^1}{e^1 - e^{-1}}$

Sub into A_1: $A_1 = 1 - \frac{e^1}{e^1 - e^{-1}} = \frac{-e^{-1}}{e^1 - e^{-1}}$

Solve for B_2: $B_2 = \frac{(r_2 - a_1)A_2}{b_2} = \frac{(-1-1)A_2}{-1} = 2A_2 = \frac{2e^1}{e^1 - e^{-1}}$

Solution: $\boxed{\begin{array}{l} x(t) = A_1 e^t + A_2 e^{-t} \\ \lambda(t) = B_2 e^{-t} \\ u(t) = -\frac{1}{2} B_2 e^{-t} \end{array}}$

where $A_1 = \dfrac{-e^{-1}}{e^1 - e^{-1}} = -0.1565$, $A_2 = \dfrac{e^1}{e^1 - e^{-1}} = 1.1565$, $B_2 = \dfrac{2e^1}{e^1 - e^{-1}} = 2.313$

2. KS II.7.1, p.162.

max $\displaystyle\int_0^1 \left(-u^2(t)/2\right)dt$

subject to $x' = y$, $y' = u$, $x(0) = 0$, $y(0) = 0$, $x(1) + y(1) \ge 2$

State Variables: x and y; Control Variable: u

$f(t,x,y,u) = -u^2/2$

$g_1(t,x,y,u) = y$

$g_2(t,x,y,u) = u$

$H \equiv f + \lambda_1 g_1 + \lambda_2 g_2 = -u^2/2 + \lambda_1 y + \lambda_2 u$

Optimal solution properties:

(1) $\dfrac{\partial H}{\partial u} = -u + \lambda_2 = 0$	(4a) $x(0) = 0$
(2a) $-\dfrac{\partial H}{\partial x} = \lambda_1'(t) = 0$	(4b) $y(0) = 0$
(2b) $-\dfrac{\partial H}{\partial y} = \lambda_2'(t) = \lambda_1$	(5) Endpoint constraint (see below)
(3a) $\dfrac{\partial H}{\partial \lambda_1} = x'(t) = y$	(6) $H_{uu} = -1 \le 0$ (required for max... good)
(3b) $\dfrac{\partial H}{\partial \lambda_2} = y'(t) = u$	

Solve (1) for u: $u = \lambda_2$

Sub into (2) and (3) to get system of differential equations... actually, u is only in (3b): $y'(t) = \lambda_2$

The normal technique of solving this problem would result in a system of four differential equations. That's too hard to solve. Fortunately, one of the equations is extremely simple and the others build off it.

Since (2a) has $\lambda_1'(t) = 0$, we must have $\lambda_1(t)$ equal to a constant: $\lambda_1(t) = c_1$

Sub into (2b): $\lambda_2'(t) = \lambda_1 = c_1 \;\Rightarrow\; \lambda_2(t) = \int \lambda_2'(t)dt = \int c_1 dt = c_1 t + c_2$

Sub into (3b): $y'(t) = \lambda_2 = c_1 t + c_2 \;\Rightarrow\; y(t) = \int y'(t)dt = \int (c_1 t + c_2)dt = \dfrac{c_1 t^2}{2} + c_2 t + c_3$

Sub into (3a): $x'(t) = y = \frac{c_1 t^2}{2} + c_2 t + c_3 \Rightarrow$

$$x(t) = \int x'(t)dt = \int \left(\frac{c_1 t^2}{2} + c_2 t + c_3 \right) dt = \frac{c_1 t^3}{6} + \frac{c_2 t^2}{2} + c_3 t + c_4$$

Use (4a): $x(0) = \frac{c_1 (0)^3}{6} + \frac{c_2 (0)^2}{2} + c_3 (0) + c_4 = 0 \Rightarrow c_4 = 0$

Use (4b): $y(0) = \frac{c_1 (0)^2}{2} + c_2 (0) + c_3 = 0 \Rightarrow c_3 = 0$

Summary - system of five equations:

[1] $x(t) = \frac{c_1 t^3}{6} + \frac{c_2 t^2}{2}$

[2] $y(t) = \frac{c_1 t^2}{2} + c_2 t$

[3] $\lambda_1(t) = c_1$

[4] $\lambda_2(t) = c_1 t + c_2$

[5] $u(t) = c_1 t + c_2$

There are two unknowns (c_1 and c_2) and now the endpoint constraints comes into play. There are two cases to consider.

<u>Case 1</u>: $x(1) + y(1) > 2$ (not binding) $\Rightarrow \lambda_1(1) = \lambda_2(1) = 0$

$\lambda_1(1) = c_1 = 0$

$\lambda_2(1) = c_1 t + c_2 = (0)t + c_2 = 0$

Makes solution pretty easy... $x(t) = y(t) = \lambda_1(t) = \lambda_2(t) = u(t) = 0$, but <u>not valid</u> because endpoint constraint is violated: $x(1) + y(1) = 0 \not> 2$

<u>Case 2</u>: $x(1) + y(1) = 2$ (binding) $\Rightarrow \lambda_1(1) = pK_x\big|_{t=1}$ and $\lambda_2(1) = pK_y\big|_{t=1}$

$$\lambda_1(1) = p\frac{\partial(x(t) + y(t))}{\partial x}\bigg|_{t=1} = p$$

$$\lambda_2(1) = p\frac{\partial(x(t) + y(t))}{\partial y}\bigg|_{t=1} = p \Rightarrow \lambda_1(1) = \lambda_2(1)$$

$\lambda_1(1) = \lambda_2(1) \Rightarrow c_1 = c_1(1) + c_2 \Rightarrow c_2 = 0$

Sub that into $x(t)$ and $y(t)$ and use the constraint:

$$x(1) + y(1) = \left(\frac{c_1 (1)^3}{6} + \frac{(0)(1)^2}{2} \right) + \left(\frac{c_1 (1)^2}{2} + (0)(1) \right) = \left(\frac{c_1}{6} \right) + \left(\frac{c_1}{2} \right) = \frac{2}{3} c_1 = 2 \Rightarrow c_1 = 3$$

Solution:
$$\boxed{\begin{aligned} x(t) &= \tfrac{1}{2} t^3 \\ y(t) &= \tfrac{3}{2} t^2 \\ \lambda_1(t) &= 3 \\ \lambda_2(t) &= u(t) = 3t \end{aligned}}$$

3. Consider the social planner's exhaustible resource problem:

$$\max_q \int_0^T e^{-rt}\left[B(q(t)) - C(q(t),t)\right]dt$$

s.t. $x' = -q(t)$, $x(0) = A$, $x(T) = 0$

where $B(q) =$ social surplus from consuming at rate q (so that $B'(q) = p(q)$, the demand price) and $C(q(t),t) = c_0 e^{-\delta t} q(t)$, $\delta > 0$ (so that unit cost [marginal and average] falls at the rate δ per unit of time).

a) Find the rate of change of price on the optimal path. Express this in terms of p_0, the initial price, c_0, the initial marginal cost, and r and δ.

b) Prove that p'/p, the rate of change of price, lies between r and $-\delta$.

State Variable: x; Control Variable: q

$f = e^{-rt}\left[B(q(t)) - c_0 e^{-\delta t} q(t)\right] = e^{-rt} B(q(t)) - c_0 e^{-(r+\delta)t} q(t)$

$g = -q(t)$

$H \equiv f + \lambda g = e^{-rt} B(q(t)) - c_0 e^{-(r+\delta)t} q(t) - \lambda q(t)$

Optimal solution properties:

(1) $\dfrac{\partial H}{\partial q} = e^{-rt} B'(q(t)) - c_0 e^{-(r+\delta)t} - \lambda = 0$ (4) $x(0) = A$

(2) $-\dfrac{\partial H}{\partial x} = \lambda'(t) = 0$ (5) $x(T) = 0$

(3) $\dfrac{\partial H}{\partial \lambda} = x'(t) = -q(t)$ (6) $H_{qq} = e^{-rt} B''(q(t))$?? 0 (no info)

Normal procedure would say to solve (1) for q, but there is no q in there

Instead substitute $B'(q) = p(q)$: $e^{-rt} p(q(t)) - c_0 e^{-(r+\delta)t} - \lambda = 0$

Solve for p: $p(q(t)) = \dfrac{c_0 e^{-(r+\delta)t} + \lambda}{e^{-rt}} = c_0 e^{-\delta t} + \lambda e^{rt}$

(2) implies λ is a constant; use initial price $p(q(0)) = p_0$ to solve for λ:

$$p(q(0)) = c_0 e^{-\delta(0)} + \lambda e^{r(0)} = c_0 + \lambda = p_0 \;\Rightarrow\; \lambda = p_0 - c_0$$

So $p(q(t)) = c_0 e^{-\delta t} + e^{rt}(p_0 - c_0)$

a) $\boxed{p' = \dfrac{\partial p}{\partial t} = -\delta c_0 e^{-\delta t} + r e^{rt}(p_0 - c_0)}$

b) $\dfrac{p'}{p} = \dfrac{-\delta c_0 e^{-\delta t} + r e^{rt}(p_0 - c_0)}{c_0 e^{-\delta t} + e^{rt}(p_0 - c_0)} =$ weighted average of r and $-\delta$ so it must lie between r and $-\delta$.

Note: weighted average is of form: $A\dfrac{x}{x+y} + B\dfrac{y}{x+y}$ ... in this case $A = -\delta$, $B = r$, $x = c_0 e^{-\delta t}$, $y = e^{rt}(p_0 - c_0)$

Documentation.

I went over all the problems with Prof Dai. For the first one, he told me to leave the answer in general form and just specify what the constants were with formulas. On 7.1 he worked through the transversality condition for the endpoint constraint... told me to solve for both cases. He also checked my logic on the functional forms and confirmed I didn't have to solve any differential equations. On the third problem, Prof Dai pointed out what the problem was asking for (i.e., didn't need to solve for $q(t)$) and told me to use $P(0) = p_0$ to solve for the constant in $P(q)$ in part a. For part b, he pointed out how p'/p is a weighted average of r and $-\delta$.